Some definitions from Lecture 2

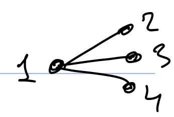
<u>Cut-norm</u> of kernel W is

$$\|W\|_\square := \sup_{S,T\subseteq[0,1]} \left| \int_{S\times T} W(x,y)\,dx\,dy \right|$$

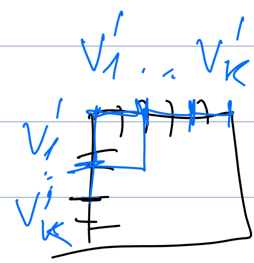
<u>Cut-distance:</u> $\delta_\square(U,W) := \inf_{\text{m.p. } \varphi:[0,1]\to[0,1]} \|U-W^\varphi\|_\square$

Graph G on $[n]$ $\rightsquigarrow$ graphon W_G

$$\begin{array}{c|cccc} & I_1 & \cdots & & I_n \\ \hline I_n & 0 & 1 & 1 & 1 \\ \vdots & 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ I_n & 1 & 0 & 0 & 0 \end{array}$$



- G on $[n]$
- graphon U with k steps



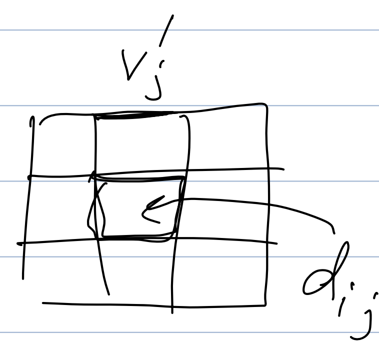
W_G

$V_i' \subseteq [0,1]$ $\Updownarrow$ $V_i \subseteq [n]$

st $\|W_G - U\|_\square$ is small

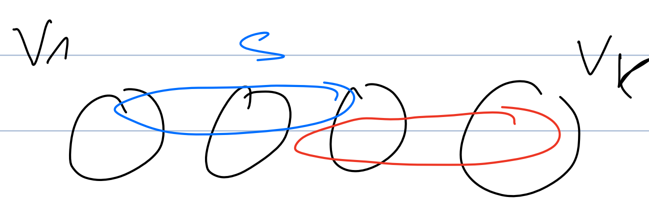
$V(G) = V_1 \cup \ldots \cup V_k$

U: v_i'



$S,T \subseteq V(G)$

$$\frac{1}{n^2} e_G(S,T) = \int_{S\times T} W_G \approx \int_{S\times T} U = \sum_{i,j=1}^{k} d_{ij}\,\frac{|V_i|}{n}\cdot\frac{|V_j|}{n}$$



Weak Regularity Partition

<u>Counting Lemma:</u> $\forall F \in \mathcal{F}_m$ $\forall U, W$

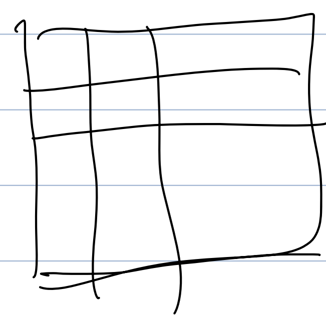
$$|t(F,W) - t(F,U)| \le \|W-U\|_\square \cdot e(F).$$

(Corollary: $\delta_\square$ conv $\Rightarrow$ density conv;
Inverse Counting Lemma: BCLSV'08)

<u>Weak Regularity lemma:</u> $\forall \varepsilon > 0\ \exists k \left(\le 4^{1/\varepsilon^2}\right)$
$\forall W$ $\exists$ k-step U s.t. $\|W-U\|_\square \le \varepsilon$

(Frieze-Kannan'99)

<u>k-step U:</u> $\exists$ partition $[0,1] = I_1 \cup \ldots \cup I_k$
s.t. U is constant on $I_i \times I_j$ $\forall ij$



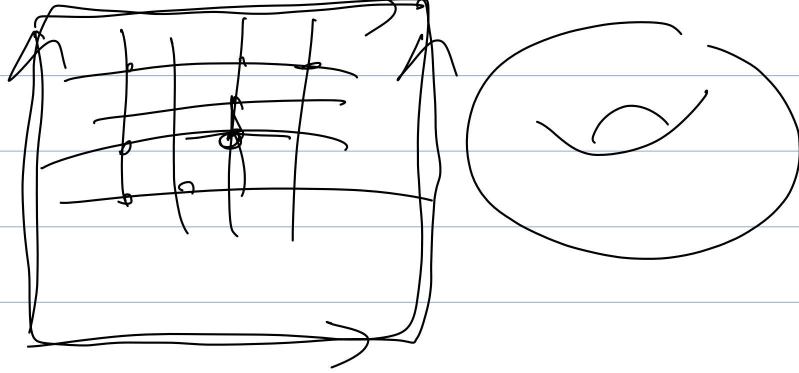
5 Bounded degree graphs

D: st $\forall$ graph has max deg $\le D$

$\mathbb{Z}^2 \subseteq \mathbb{R}^2$

$\mathbb{Z}^3 \subseteq \mathbb{R}^3$

(Statistical Physics)



Graphon limit is 0

$\boxed{\text{if disconnected: } \hom(F \sqcup H, G) = \hom(F,G)\cdot\hom(H,G)}$

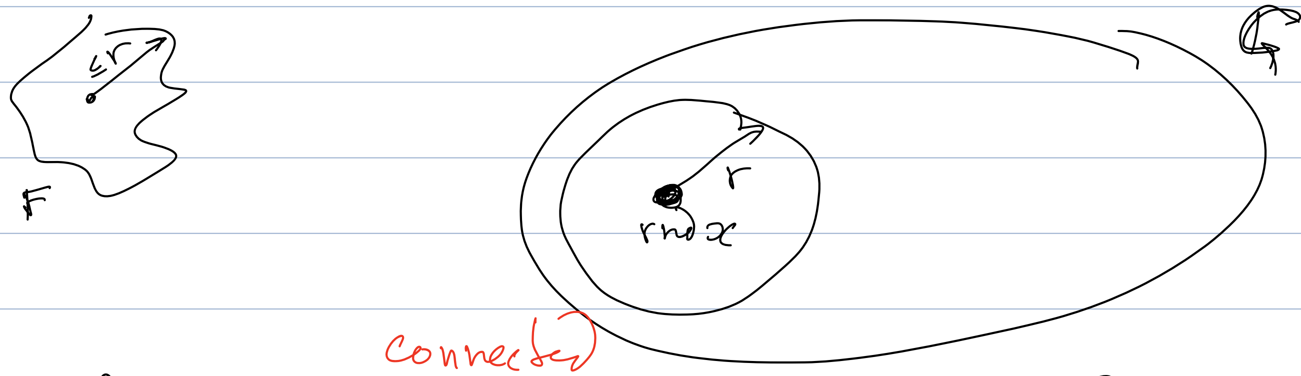
$\hom(F, G)/v(G) =: t^*(F,G)$

connected

(G_n) <u>locally converges</u> if $\forall$ connected F

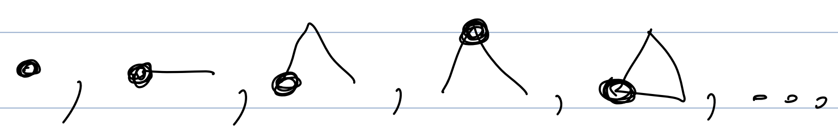
$t^*(F, G_n)$ converges

Universe is $\mathcal{G} = \{H : \Delta(H) \le D\}$



$\mathcal{G}^1 := \{(G, x) :$ connected graph $G \in \mathcal{G},\ x \in V(G)\}$

$= \{$connected rooted graphs$\}$ (up to iso)

$= \{$  $\}$

$B_{(G,x),r} = \left(G[\{y : \mathrm{dist}_G(x,y) \le r\}], x\right) \in \mathcal{G}^1$

(r-Ball)

Prob meas on $\mathcal{G}^1$

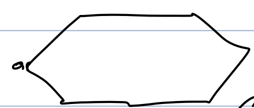
r-Ball sample $\rho_{G,r}$:

- uniform $x \in V(G)$
- output $B_{(G,x),r}$

Ex $\rho_{C_n,r} :=$

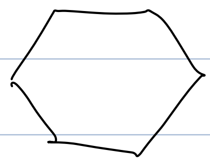
P_{2r+1}

, if $2r+1 < n$



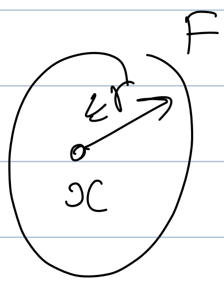
C_n o/w

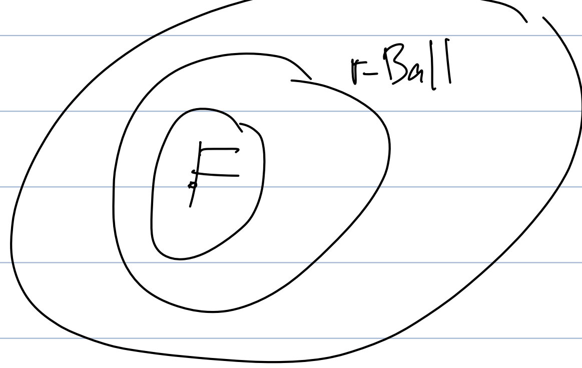
C_n:



Lemma (G_n) is locally convergent iff $\forall r$ $\rho_{G_n,r}$ converges (to some ρ_r).

$|\mathrm{supp}| \le (D+1)^r$

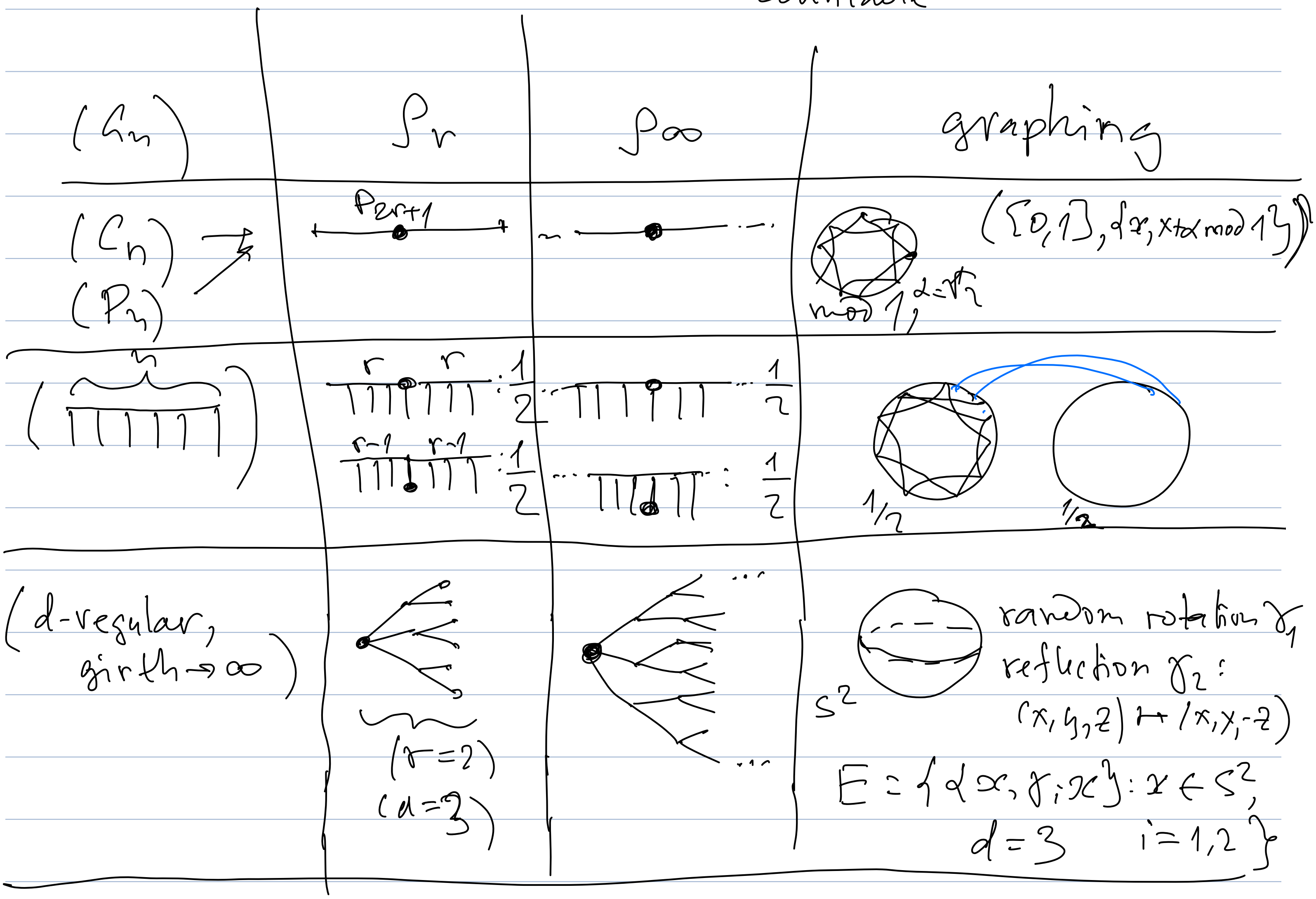




ρ_{r+1} determines ρ_r $\rightsquigarrow$ ρ_∞

$\rightsquigarrow \rho_\infty \in \mathrm{Prob}(\{\text{connected } G \text{ with } \Delta(G) \le D\})$

countable

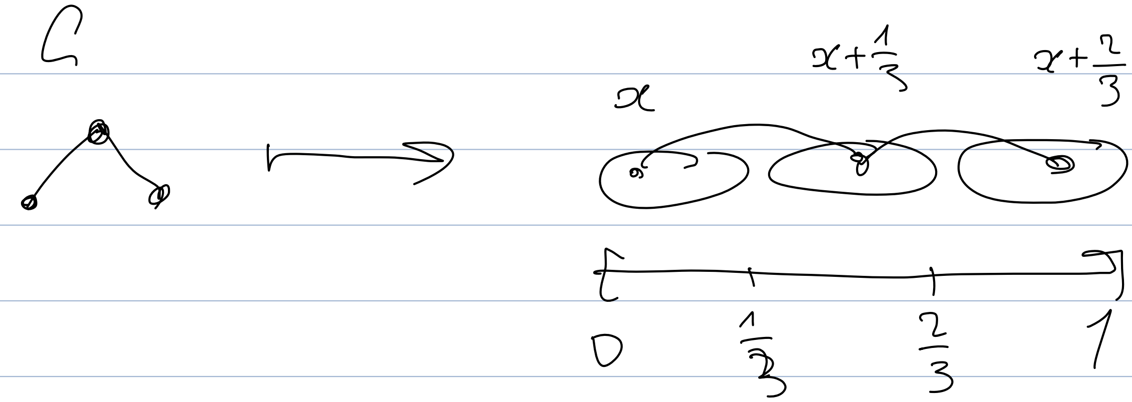


Graphing: $(V, E, \mathcal{B}, \mu) =: H$

- V: vertex set (can assume $[0,1]$)
- $(V, \mathcal{B}, \mu)$: standard prob. space ($([0,1], \{\text{Borel}\}, \text{Lebesgue measure})$)
- (V, E) is m.p. graph (wrt μ) with $\Delta \le D$

$\rho_r(H)$:
- sample $x \in V$, $x \sim \mu$
- output $B_{((V,E),x),r}$

Finite G



$\forall$ loc. conv (G_n) $\exists$ graphing which is its local limit
(Aldous-Lyons, Elek 2007)

(G_n) locally conv to a graphing H if

$\forall r \quad \rho_{G_n,r} \longrightarrow \rho_{H,r}$

$\rho_\infty(H)$: sample $x \in V(H)$ & output the whole component of x (a rooted countable graph)