Homework for Lecture 2

Note that, since kernels and graphons were not discussed at Lecture 1, you can submit your solution to Problem 5 with this assignment.

Problem 6 Let G and H be two graphs such that for every graph F it holds that $\hom(F, G) = \hom(F, H)$. Prove that G and H are isomorphic to each other.

[Remark: With a bit more work, this result can be used to show that the map that sends a graph G to $\phi_G \in$ LIM is injective apart of identifying blowups.]

Problem 7 Give an example of graphons $W, W_1, W_2, W_3, \ldots$ such that $\|W - W_n\|_\Box \to 0$ but W_n does not converge to W in the L^1-norm (that is, $\int_{[0,1]^2} |W_n(x, y) - W(x, y)| \,\mathrm{d}x\, \mathrm{d}y \not\to 0$) as $n \to \infty$.

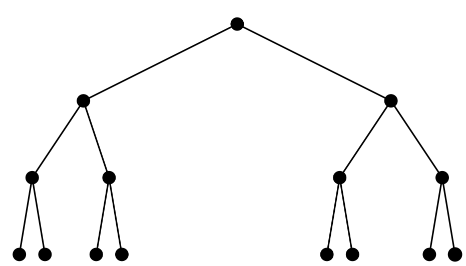
[You do not need to prove that your example has these properties; an informal justification is perfectly ok for this problem.]

Problem 8 Prove that for all graphs F, G and every vertex $x \in V(F)$ is holds that $t^*(F, G) := \hom(F, G)/v(G)$ is the average over $y \in V(G)$ of $\hom((F, x), (G, y))$ which we define to be the number of homomorphisms from F to G that map x to y.

[Remark: In particular, this average does not depend on the choice of $x \in V(F)$; also, it follows that if every vertex of F is at distance at most r from x then the distribution $\rho_r(G)$ of r-balls in G determines $t^*(F, G)$.]

Problem 9 Express the k-th moment $M_k(G) := \frac{1}{v(G)} \sum_{v \in V(G)} (\deg(v))^k$ of the degree sequence of a graph G in terms of some scaled homomorphism densities $t^*(\cdot, G)$.

Problem 10 Let B_n be the *complete depth-n binary tree* which is the graph on binary sequences of length at most n where two sequences are adjacent if one can be obtained from the other by removing the last symbol. Thus B_n is a tree with $1 + 2 + \ldots + 2^n = 2^{n+1} - 1$ vertices (as we allow the empty sequence); here is a drawing of B_3:



Describe the limiting distribution of r-balls for the sequence $(B_n)_{n\in\mathbb{N}}$ when $r = 1$ and $r = 2$ as $n \to \infty$.

[Optional: Describe a graphing which is a local limit of (B_n).]