Poisson Summation formula:

For $f \in \mathcal{S}(\mathbb{R}^d)$ (Schwartz space)

$$\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{m \in \mathbb{Z}^d} \hat{f}(m).$$

where $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x \cdot \xi}\, dx.$

Pf: (for $d=1$) Let $F(x) := \sum_{n \in \mathbb{Z}} f(x+n)$.

- F is 1-periodic, and smooth, (say, C^2)
- Fourier series converges uniformly:

$$F(x) = \sum_{m \in \mathbb{Z}} a_m e^{-2\pi i x m} \quad \text{and:}$$

$$a_m = \int_0^1 F(x) e^{-2\pi i m x}\, dx$$

decay of f

$$= \sum_{n \in \mathbb{Z}} \int_0^1 f(x+n) e^{-2\pi i m x}\, dx$$

$$= \sum_{n \in \mathbb{Z}} \int_0^1 f(x+n) e^{-2\pi i m (x+n)}\, dx = \hat{f}(m)$$

$\square$

Remarks: • Suffice to assume $|f(x)| \lesssim (1+|x|)^{-a}$, $|\hat{f}(\xi)| \lesssim (1+|\xi|)^{-a}$, $a>d$.

• Extends to lattices:

$$\sqrt{\mathrm{Vol}(\mathbb{R}^d/\Gamma)} \sum_{r\in\Gamma} f(r) = \sqrt{\mathrm{Vol}(\mathbb{R}^d/\Gamma^*)} \sum_{r'\in\Gamma^*} \hat{f}(r')$$

$$\Gamma^* = \{ m \in \mathbb{R}^d : \langle m, n\rangle \in \mathbb{Z} \ \forall n \in \Gamma \}.$$

Applications:

(1) Theta function $\Theta(t) := \sum_{m\in\mathbb{Z}} e^{-\pi m^2 t}$. $t>0$

We have:

$$\widehat{\exp(-\pi a^2 (\cdot)^2)}(\xi) = \frac{1}{a} \exp\left(-\frac{\pi}{a^2}\xi^2\right)$$

$$\Rightarrow \quad \boxed{\Theta\left(\frac{1}{t}\right) = \sqrt{t}\,\Theta(t).}$$

In particular, since $\Theta(s) \xrightarrow{s\to\infty} 1$ we get

$$\Theta(t) \sim \frac{1}{\sqrt{t}} \quad t \to 0^+.$$

Whittaker - Shannon

② **Sampling Thm:** Let $f \in L^2(\mathbb{R})$, $\operatorname{spt}(\hat{f}) \subset [-\frac{1}{2}, \frac{1}{2}]$

"band - limited"

Then:

$$f(x) = \sum_{k \in \mathbb{Z}} f(k) \frac{\sin(\pi(x-k))}{\pi(x-k)}$$

$$= \frac{\sin(\pi x)}{\pi} \sum_{k \in \mathbb{Z}} (-1)^k \frac{f(k)}{x-k} \quad \text{in } L^2(\mathbb{R}).$$

Pf: We have:

$$\hat{f}(\xi) = \left(\sum_{m \in \mathbb{Z}} \hat{f}(\xi + m) \right) 1_{[-\frac{1}{2}, \frac{1}{2}]}(\xi)$$

$$\overset{(PSF)}{=} 1_{[-\frac{1}{2}, \frac{1}{2}]}(\xi) \sum_{m \in \mathbb{Z}} f(m) e^{2\pi i \xi \cdot m}$$

Using Inversion, we get:

$$f(x) \overset{L^2}{=} \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} d\xi = \int_{-1/2}^{1/2} \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

$$\text{"="} \sum_{m \in \mathbb{Z}} f(m) \int_{-1/2}^{1/2} e^{2\pi i \xi (x-m)} d\xi.$$

How to justify "="?

$$\int_{-1/2}^{1/2} \left| \hat{f}(\xi) - \sum_{|m|\leq N} f(m) e^{2\pi i \xi \cdot m} \right|^2 d\xi \xrightarrow{N \to \infty} 0$$

By L^2-convergence of Fourier series.

In particular:

$$\left\| f - \sum_{|m|\leq N} f(m) \frac{\sin(\pi(\cdot - m))}{\pi(\cdot - m)} \right\|_{L^2(\mathbb{R})}$$

$$\leq \left\| \int_{-1/2}^{1/2} \left(\hat{f}(\xi) - \sum_{|m|\leq N} f(m) e^{2\pi i \xi \cdot m} \right) e^{2\pi i \xi \cdot} d\xi \right\|_{L^2(\mathbb{R})}$$

Parseval

$$= \int_{\mathbb{R}} 1_{[-1/2, 1/2]}(\xi) \left| \hat{f}(\xi) - \sum_{|m|\leq N} f(m) e^{2\pi i \xi \cdot m} \right|^2 d\xi \xrightarrow{N \to \infty} 0$$

$\square$

Remarks:

- Can prove local unif. convergence in $\mathbb{C}$.

  Hint: use smooth cut-off of $1_{[-1/2, 1/2]}$.

- $PW_{1/2} := \{ f \in L^2(\mathbb{R}) \mid \operatorname{spt}(\hat{f}) \subset [-1/2, 1/2] \}$ is closed sub-space of $L^2(\mathbb{R})$.

(3) Minkowski's Thm: $K \subset \mathbb{R}^d$ · closed · convex · symmetric $K = -K$.

If $\mathrm{Vol}(K) \geq 2^d \implies \exists w \in \mathbb{Z}^d \setminus \{0\}$ s.t $w \in K$.

(This is sharp $K_\varepsilon = [1-\varepsilon, 1+\varepsilon]^n$).

Pf: Let $f = 1_K$. It is even and real

$\implies \hat{f}$ is even and real.

By PSF:

$$2^d \sum_{n \in \mathbb{Z}^d} (f * f)(2n) = \sum_{m \in \mathbb{Z}^d} \left| \hat{f}\left(\frac{m}{2}\right)\right|^2 \quad (*)$$

Suppose that for some $n \in \mathbb{Z}^d$,

$$0 \neq (f * f)(2n) = \int_{\mathbb{R}^d} 1_K(2n - x) 1_K(x) \, dx.$$

then for some $x \in \mathbb{R}^d$ we have:

$$2n - x, x \in K \implies \tfrac{1}{2}(2n - x) + \tfrac{1}{2}x \in K$$

$$\implies n \in K.$$

By assumption, $n = 0$.

$$\text{LHS of } (*) = 2^n \cdot (f * f)(0) = 2^n \cdot \int_{\mathbb{R}^d} f^2 \, dx$$

$$= 2^n \cdot \mathrm{Vol}(K).$$

On the RHS:

$$\sum_{m \in \mathbb{Z}^d} \left| \hat{f}\left(\tfrac{m}{2}\right) \right|^2 > |\hat{f}(0)|^2 = \mathrm{Vol}(K)^2.$$

$$\sum_{m \neq 0} \left| \hat{f}\left(\tfrac{m}{2}\right) \right|^2 = 0 \overset{\text{PSF}}{\Longleftrightarrow} \sum_{n \in \mathbb{Z}^d} f(x + 2n) \text{ is const.}$$

but $f(0) = 1$ and f vanish at closest integer pt.

We get from $(*)$:

$$\mathrm{Vol}(K) < 2^d$$

$\square$

<u>Remark:</u> We cheated! $f \notin \mathcal{S}(\mathbb{R}^d)$

Correct via approximations ....

Wirtinger's inequality: Let $f \in C^1[0,1]$.

- $f(0) = f(1)$, $\int_0^1 f = 0 \implies \int_0^1 |f|^2 \le \frac{1}{(2\pi)^2} \int_0^1 |f'|^2$.

- $f(0) = f(1) = 0 \implies \int_0^1 |f|^2 \le \frac{1}{\pi^2} \int_0^1 |f'|^2$

[Both are sharp: $\sin(2\pi x)$, $\sin(\pi x)$]

Pf: For the first inequality, recall that:

- $$\widehat{f'}(n) = (2\pi i n) \cdot \hat{f}(n) \qquad \text{(IBP)}$$

Since $\hat{f}(0) = \int_0^1 f = 0$, Parseval formula:

$$\int_0^1 |f|^2 = \sum_{n \neq 0} |\hat{f}(n)|^2 = \sum_{n \neq 0} \frac{1}{(2\pi)^2} \frac{1}{|n|^2} |\widehat{f'}(n)|^2$$

$$\le \frac{1}{(2\pi)^2} \sum_{n \in \mathbb{Z}} |\widehat{f'}(n)|^2 = \frac{1}{(2\pi)^2} \int_0^1 |f'|^2 .$$

For the second ineq., extend f to an odd 2-periodic func. $\hat{f}$, $g(t) = \hat{f}(2t)$.

Apply prev. ineq. and get:

$$4\pi^2 \int_0^1 |f|^2 = 4\pi^2 \int_0^1 |g|^2 \leq \int_0^1 |g'|^2 = 4 \int_0^1 |f'|^2 .$$

$\square$

<u>Stable Wirtinger:</u> $f \in C^1[a,b]$, $\varepsilon > 0$.

$$\int_a^b |f|^2 \leq (1+\varepsilon)\left(\frac{b-a}{\pi}\right)^2 \int_a^b |f'|^2 + \left(1+\frac{1}{\varepsilon}\right)(b-a)\left(|f(a)|^2 + |f(b)|^2\right).$$

<u>Pf:</u>