Lecture 2

I. Borell's thm ⇒ Brunn-Minkowski inequality

- Lebesgue measure μ has density $f(x) = 1$

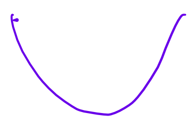
  f is log-concave $1 = e^{-0}$, 0 convex function

  ⇒ Lebesgue measure satisfies BM.

- Gaussian measure $d\gamma = e^{-\frac{|x|^2}{2}} \cdot (2\pi)^{-n/2} dx$

  $f(x) = e^{-\frac{|x|^2}{2}} \cdot c_n = e^{-V}$

  $V(x) = \frac{|x|^2}{2} + \tilde{c}_n$ – convex function



$$\gamma(\lambda K + (1-\lambda)L) \geq \gamma(K)^{\lambda} \gamma(L)^{1-\lambda}$$

– info about normal distribution

Theorem (Prékopa-Leindler)

$\lambda \in [0,1]$, $f, g, h : \mathbb{R}^n \to \mathbb{R}$ (measurable) s.t. $\int e^{-f}, \int e^{-g}, \int e^{-h}$ all exist

s.t. $\forall x, y \in \mathbb{R}^n$ $\quad h(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) g(y)$

THEN

$$\int e^{-h} \geq \left(\int e^{-f}\right)^{\lambda} \cdot \left(\int e^{-g}\right)^{1-\lambda}$$

Derivation of Borell's thm (and thus Brunn-Minkowski) from Prekopa-Leindler inequality

$d\mu = \varphi(x)\,dx$ — measure with density φ

φ is log-concave

K, L — Borell measurable

$$e^{-h(x)} = \varphi(x) \cdot \mathbb{1}_{\lambda K + (1-\lambda)L}(x)$$

$$e^{-f(x)} = \varphi(x) \cdot \mathbb{1}_K(x)$$

$$e^{-g(x)} = \varphi(x) \cdot \mathbb{1}_L(x)$$

note: $h(\lambda x + (1-\lambda) y) \le \lambda f(x) + (1-\lambda) g(y)$ (*)

due to log-concavity of φ

Prekopa condition is satisfied $\Rightarrow$

$$\int e^{-h} \ge \left(\int e^{-f}\right)^{\lambda}\left(\int e^{-g}\right)^{1-\lambda}$$

$$\int_{\lambda K + (1-\lambda)L} \varphi \ge \left(\int_K \varphi\right)^{\lambda}\left(\int_L \varphi\right)^{1-\lambda}$$

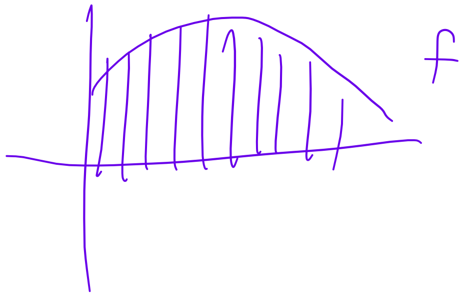
$$\mu(\lambda K + (1-\lambda)L) \ge \mu(K)^{\lambda} \cdot \mu(L)^{1-\lambda}$$ ☒

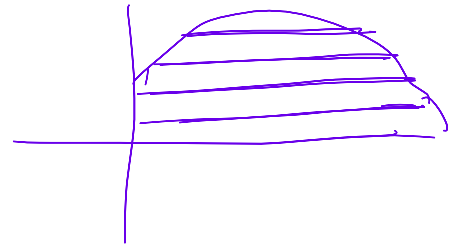
Proof of the Prekopa-Leindler inequality

By induction in dimension

Recall the "layer-cake formula" $f \ge 0$

$$\int_{\mathbb{R}^n} f(x)\,d\mu = \int_0^{\infty} \mu(\{x\in\mathbb{R}^n : f(x) > t\})\,dt$$





$$F(x) = e^{-f(x)} \geq 0$$

$$G(x) = e^{-g(x)} \geq 0$$

$$H(x) = e^{-h(x)} \geq 0$$

$$h(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) g(y)$$

$$H(\lambda x + (1-\lambda) y) \geq F(x)^{\lambda} G(y)^{1-\lambda}$$

. we want $\int H \geq \left(\int F\right)^{\lambda} \left(\int G\right)^{1-\lambda}$

<u>step 1</u> dimension = 1

$$\int_{\mathbb{R}} H = \int_0^{\infty} |\{t\in\mathbb{R} : H(t) > s\}|\,ds \ \textcircled{\geq}$$

logical exercise

$$\{H>s\} \supset \lambda\{F>s\} + (1-\lambda)\{G>s\}$$

↑ 1-dim Minkowski sum

$$\textcircled{\geq} \int_0^{\infty} \left(\lambda\,|\{F>s\}| + (1-\lambda)\,|\{G>s\}|\right) ds$$

$$= \lambda \int_0^{\infty} |\{F>s\}|\,ds + (1-\lambda) \int_0^{\infty} |\{G>s\}|\,ds$$

$$= \lambda \int_{\mathbb{R}} F + (1-\lambda) \int_{\mathbb{R}} G \geq \left(\int_{\mathbb{R}} F\right)^{\lambda} \cdot \left(\int_{\mathbb{R}} G\right)^{1-\lambda}$$

↑ layer-cake

$\lambda a + (1-\lambda) b \geq a^{\lambda} b^{1-\lambda}$

Base of induction - verified

Step 2 induction

Suppose the P-L thm is verified in $\mathbb{R}^{n-1}$.

$$H_n(x_n) = \int_{\mathbb{R}^{n-1}} H(x_1, \dots, x_n)\, dx_1 \dots dx_{n-1}$$

$$x = (x_1 \dots x_n) = (\bar{x}, x_n)$$

Same way F_n, G_n.

We have $H(\lambda(\bar{x}, x_n) + (1-\lambda)(\bar{y}, y_n)) \geq F((\bar{x}, x_n))^{\lambda} \cdot G((\bar{y}, y_n))^{1-\lambda}$ ✓

by the assumption of P-L

induction hypothesis: on $\mathbb{R}^{n-1}$ P-L holds

$$\int_{\mathbb{R}^{n-1}} H(x_1 \dots x_n) \geq \left(\int_{\mathbb{R}^{n-1}} F\right)^{\lambda} \left(\int_{\mathbb{R}^{n-1}} G\right)^{1-\lambda} \quad \text{i.e.}$$

$$H_n(\lambda x_n + (1-\lambda) y_n) \geq F_n(x_n)^{\lambda} G_n(y_n)^{1-\lambda}$$

(1-dim functional functions)

By step 1 $\quad \int H_n \geq \left(\int F_n\right)^{\lambda} \left(\int G_n\right)^{1-\lambda}$

i.e. $\quad \int H \geq \left(\int F\right)^{\lambda} \left(\int G\right)^{1-\lambda}$

∎

Thus the Brunn-Minkowski Inequality is also proven! So is the isoperimetric Inequality!

Remark Prékopa-Leindler inequality is a member of a larger family, Brascamp-Lieb inequality check notes

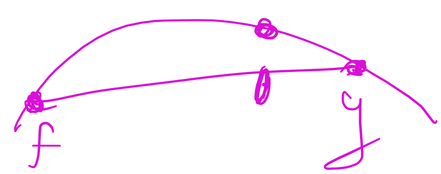
(II) Main idea: From concavity principles to isoperimetry via linearizations.

concavity principle

In general, suppose $\mathcal{F}$ is a functional

$$\forall f, g \quad \mathcal{F}((1-t)f + tg) \geq (1-t)\mathcal{F}(f) + t\mathcal{F}(g)$$

$\forall t \in [0,1]$ (i.e. $\mathcal{F}$ is concave)



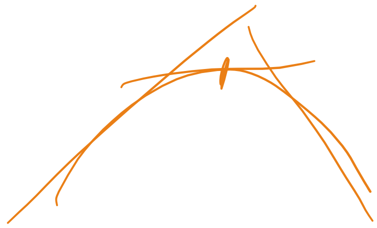
Define

$$\alpha(t) = \mathcal{F}((1-t)f + tg) - (1-t)\mathcal{F}(f) - t\mathcal{F}(g)$$

- $\alpha(t) \geq 0$ on $t \in [0,1]$
- $\alpha(0) = 0$

① If $\mathcal{L}'(0)$ makes sence then

$$\mathcal{L}''(0) \geq 0$$



F - convex and twice differentiable

$F'' \leq 0$

$F(x) = -x^2$

$F'' = -2$

② $$\mathcal{L}''(0) \leq 0$$

$\frac{d^2}{dt^2}\mathcal{F}$

Concavity principles give new inequalities!

Example $\log|\lambda K + (1-\lambda)L| \geq \lambda \log|K| + (1-\lambda)\log|L|$

$\mathcal{F} = \log|K|$

$\mathcal{L}(t) = \log|tK + (1-t)L| - t\log|K| - (1-t)\log|L|$

$$\mathcal{L}'(0) \geq 0$$

$K = B_2^n$ this amounts to isoperimetric inequality

In general, Minkowski's first inequality:

K L convex bodies

$$V_1(K,L) = \liminf_{\varepsilon \to 0} \frac{|K+\varepsilon L \setminus K|}{\varepsilon} \cdot \frac{1}{n}$$

"anisotropic perimeter"

$$V_1(K,L) \geq |K|^{\frac{n-1}{n}} \cdot |L|^{1/n}$$

(III) How to understand the P-L inequality as a concavity principle?

$$h(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) g(y) \quad (\star)$$

$$\Rightarrow \int e^{-h} \geq \left(\int e^{-f}\right)^{\lambda} \cdot \left(\int e^{-g}\right)^{1-\lambda}$$

Suppose f, g are log-concave

What is the optimal h to satisfy $(\star)$? $\quad \lambda = 1-t$

Def "inf-convolution" $\quad t \in [0,1]$

$$f \square_t g \,(z) = \inf_{(1-t)x + ty = z} \big((1-t) f(x) + t g(y)\big)$$

function that interpolates between f and g $\quad t \in [0,1]$

- $f \square_t g\,((1-t)x + ty) \geq (1-t) f(x) + t g(y)$
- $f \square_t g$ is smallest possible to satisfy it

Prop: $\int e^{-f \square_t g} \ldots \left(\int e^{-f}\right)^{t} / \left(\int e^{-g}\right)^{1-t}$

$P\text{-}L: \quad \int e \quad \geq (\int e)(\int e)$

$f \,\square_0\, g = f$

$f \,\square_1\, g = g$

want to find a transform T s.t.

$T\big((1-t)f + tg\big) = f \,\square_t\, g$?

<u>Example</u> $\quad f(x) = \mathbb{1}_K^\infty(x) = \begin{cases} 0 & \text{if } x \in K \\ \infty & \text{if } x \notin K \end{cases}$

K - convex set

$\mathbb{1}_K^\infty \,\square_{1/2}\, \mathbb{1}_L^\infty = \inf\limits_{x,y:\ \frac{x+y}{2} = z} \left(\dfrac{\mathbb{1}_K^\infty(x) + \mathbb{1}_L^\infty(y)}{2} \right)$

$\stackrel{\circ}{=} \mathbb{1}^\infty_{\frac{K+L}{2}}$

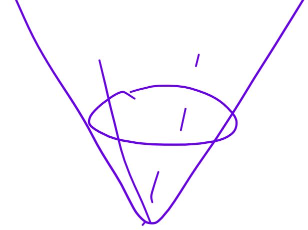
<u>Def</u> (support function of a convex body K)

$h_K(x) = \sup\limits_{y \in K} \langle x, y \rangle = \|x\|_{K^0}$

(if K is symmetric)

h ... convex function ...

- h_K convex function



- $h_{K+L} = h_K + h_L$

i.e. we want $\mathcal{T}\, \mathbb{1}_K^{\infty} = h_K$?

IV Legendre transform

Def $f: \mathbb{R}^n \to \mathbb{R}$ function

$$f^*(x) := \sup_{y \in \mathbb{R}^n} \left(\langle x, y \rangle - f(y) \right)$$

Claim $(f \,\square_t\, g)^* = (1-t) f^* + t g^*$

Reformulation of the Prékopa-Leindler inequality

$$\int e^{-((1-t)f + tg)^*} \geq \left(\int e^{-f^*} \right)^t \left(\int e^{-g^*} \right)^{1-t}$$

i.e. $\log \int e^{-f^*}$ is concave !!!

Remark Hölder's inequality $\log \int e^{-f}$ is convex