Summer school “Geometric and functional inequalities, linearizations and isoperimetry”

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Abstract

In this mini-course, we discuss isoperimetric-type inequalities related to log-concave measures (and, in particular, convex bodies). We interpret several classical inequalities as concavity principles and employ the powerful idea of linearization to understand further isoperimetric-type questions about functions. We will discuss inequalities, such as the Prekopa-Leindler inequality, a Generalized form of the Log-Sobolev inequality, the functional Ehrhard inequality, the Brascamp-Lieb inequality and Bobkov’s inequality, as well as the phenomenon of the Gaussian isoperimetry. Several approaches to these topics will be discussed and some novel variants will be outlined as well.

Disclaimer: these lecture notes are currently under construction and may not be fully proofread yet. If you spot a typo, please let me know!

Contents

Notation and preliminaries from Linear Algebra and geometry of $\mathbb{R}^n$

Notation

- $\mathbb{R}^n$ – the $n-$dimensional space
- $|\cdot|$ or $|\cdot|_k$ (for a set) – Lebesgue $k-$dimensional volume
- $\langle\cdot,\cdot\rangle$ – the scalar product in $\mathbb{R}^n$.
- For $x \in \mathbb{R}^n$, denote $x^2 = \langle x, x\rangle = |x|^2$.
- B_2^n – the Euclidean ball

- $\mathbb{S}^{n-1}$ – the unit sphere
- $A + B$ – Minkowski sum of sets
- For a vector $x \in \mathbb{R}^n$, $\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$ is the p-norm in $\mathbb{R}^n$ for $p \geq 1$
- $\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$ is the ∞-norm
- $|x| = \|x\|_2$ is shorthand for Euclidean length
- $B_p^n = \{x \in \mathbb{R}^n \colon \|x\|_p \leq 1\}$ is the p-ball in $\mathbb{R}^n$
- For $\theta \in \mathbb{R}^n \setminus \{0\}$, we may consider the hyperplane

  $$\theta^\perp = \{x \in \mathbb{R}^n \colon \langle x, \theta \rangle = 0\},$$

  and the affine hyperplane

  $$\theta^\perp + t\theta = \{x \in \mathbb{R}^n \colon \langle x, \theta \rangle = t\}$$

  for all $t \in \mathbb{R}$.
- A half-space is a set of the form $\{x \in \mathbb{R}^n : \langle x, \theta \rangle \leq t\}$, for some given $\theta \in \mathbb{R}^n \setminus \{0\}$ and $t \in \mathbb{R}$.
- A strip is a set of the form $\{x \in \mathbb{R}^n : |\langle x - y, \theta \rangle| \leq t\}$, for some given $y \in \mathbb{R}^n$, $\theta \in \mathbb{R}^n \setminus \{0\}$ and $t \geq 0$.
- Fix $x \in \mathbb{R}$. Then $[x]$ is the floor function, the largest integer which is no larger than x.
- Recall that volume in $\mathbb{R}^n$ is $n-$homogeneous, i.e. $|tA| = t^n|A|$ for any Borel-measurable set A in $\mathbb{R}^n$ and any $t > 0$.

1 Brunn-Minkowski inequality and related concavity principles

Many results related to convexity, high-dimensional analysis, concentration of measure, geometry, probability, and other areas are intimately connected to the following fundamental result. It will be the cornerstone of this section and is undoubtedly one of the most important results in this course. Much has been written about this inequality; we particularly recommend the works of Artstein-Avidan, Giannopoulos, and Milman [1], Gardner [**Ga**], and Schneider [31].

1.1 Brunn–Minkowski Inequality and the Isoperimetric Inequality

We begin by formulating this central result:

Theorem 1.1 (Brunn–Minkowski inequality)**.** *Let $K, L \subseteq \mathbb{R}^n$ be Borel-measurable sets. Then*

$$|K + L|^{1/n} \geq |K|^{1/n} + |L|^{1/n},$$

where the Minkowski sum of K and L is defined as

$$K + L = \{x + y : x \in K,\ y \in L\}.$$

Remark 1.1. *We assume Borel measurability of K and L to ensure that the set $K + L$ is also measurable. Lebesgue measurability alone would not suffice for this conclusion; we leave both technical details as an exercise for readers particularly interested in measure theory.*

Note that for $\lambda > 0$, we have $|\lambda K| = \lambda^n |K|$. Hence, the theorem is equivalent to the statement that for all $\lambda \in [0, 1]$,

$$|\lambda K + (1 - \lambda)L|^{1/n} \geq \lambda |K|^{1/n} + (1 - \lambda)|L|^{1/n}. \tag{1}$$

In other words, the function $K \mapsto |K|^{1/n}$ is *concave* under Minkowski addition. We will soon see that many other functionals exhibit similar concavity or convexity properties.

Note that for all $a, b \geq 0$, $\lambda \in [0, 1]$, and $p > 0$,

$$(\lambda a^p + (1 - \lambda) b^p)^{1/p} \geq a^\lambda b^{1-\lambda}.$$

Thus, taking $a = |K|$ and $b = |L|$, Theorem 1.1 (equivalently, inequality (1)) implies a dimension-free version of the Brunn–Minkowski inequality:

$$|\lambda K + (1 - \lambda)L| \geq |K|^\lambda |L|^{1-\lambda}, \quad \text{for all } \lambda \in [0, 1]. \tag{2}$$

In fact, the validity of (2) for all λ implies the additive form of the Brunn–Minkowski inequality (Theorem 1.1). This implication uses the homogeneity of Lebesgue measure and is left as an exercise.

We are now ready to prove the following:

Theorem 1.2 (the Isoperimetric inequality)**.** *For all $K \subseteq \mathbb{R}^n$,*

$$\frac{|\partial K|_{n-1}}{|K|^{\frac{n-1}{n}}} \geq \frac{|\partial B_2^n|_{n-1}}{|B_2^n|^{\frac{n-1}{n}}}.$$

Proof (using the Brunn–Minkowski inequality). We outline, using the Brunn-Minkowski inequality along with the $n-$homogeneity of the volume in $\mathbb{R}^n$ and the Newton's binomial:

$$\begin{aligned}
|\partial K|_{n-1} &= \liminf_{\varepsilon\to 0} \frac{|K+\varepsilon B_2^n| - |K|}{\varepsilon} \\
&\geq \liminf_{\varepsilon\to 0} \frac{(|K|^{1/n} + |\varepsilon B_2^n|^{1/n})^n - |K|}{\varepsilon} \\
&= \liminf_{\varepsilon\to 0} \frac{(|K|^{1/n} + \varepsilon|B_2^n|^{1/n})^n - |K|}{\varepsilon} \\
&= \liminf_{\varepsilon\to 0} \frac{|K| + n|K|^{\frac{n-1}{n}}\varepsilon|B_2^n|^{1/n} + O(\varepsilon^2) - |K|}{\varepsilon} \\
&= n|K|^{\frac{n-1}{n}}|B_2^n|^{1/n}.
\end{aligned}$$

Rearranging gives:

$$\frac{|\partial K|_{n-1}}{|K|^{\frac{n-1}{n}}} \geq n|B_2^n|^{1/n} = \frac{|\partial B_2^n|_{n-1}}{|B_2^n|^{\frac{n-1}{n}}},$$

where in the last equation we used the fact that

$$|\mathbb{S}^{n-1}|_{n-1} = n|B_2^n|_n,$$

which follows from the polar coordinate integration:

$$|B_2^n|_n = \int_{\mathbb{S}^{n-1}} \int_0^1 t^{n-1} dt d\theta = |\mathbb{S}^{n-1}|_{n-1} \int_0^1 t^{n-1} dt,$$

and it remains to note that $\int_0^1 t^{n-1}dt = \frac{1}{n}$. $\square$

We note that the equality in the Brunn–Minkowski inequality holds:

$$|K+L|^{1/n} = |K|^{1/n} + |L|^{1/n}$$

if and only if $K = tL + v$ for some $t \geq 0$ and $v \in \mathbb{R}^n$. See [31] for a proof of this equality case characterization.

1.2 Proof of the Brunn–Minkowski Inequality (general case, due to Lazar Lyusternik, 1935)

Following the early works of Brunn and Minkowski, the inequality was proven in full generality by Lyusternik in 1935. Subsequent proofs were given by Henstock and Macbeath [**HenMac**], and by Hadwiger and Ohmann [17]. We now present this classical proof.

Step 1. Suppose K and L are coordinate boxes, i.e.,

$$K = [0, a_1] \times \cdots \times [0, a_n], \quad L = [0, b_1] \times \cdots \times [0, b_n].$$

Then

$$K + L = [0, a_1 + b_1] \times \cdots \times [0, a_n + b_n],$$

and thus

$$|K + L| = \prod_{i=1}^{n} (a_i + b_i).$$

Note, from the AM–GM inequality:

$$\prod_{i=1}^{n} \left(\frac{a_i}{a_i + b_i} \right)^{1/n} + \prod_{i=1}^{n} \left(\frac{b_i}{a_i + b_i} \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} \frac{a_i}{a_i + b_i} + \frac{1}{n} \sum_{i=1}^{n} \frac{b_i}{a_i + b_i} = 1.$$

and therefore,

$$\prod_{i=1}^{n} (a_i + b_i)^{1/n} \geq \prod_{i=1}^{n} a_i^{1/n} + \prod_{i=1}^{n} b_i^{1/n},$$

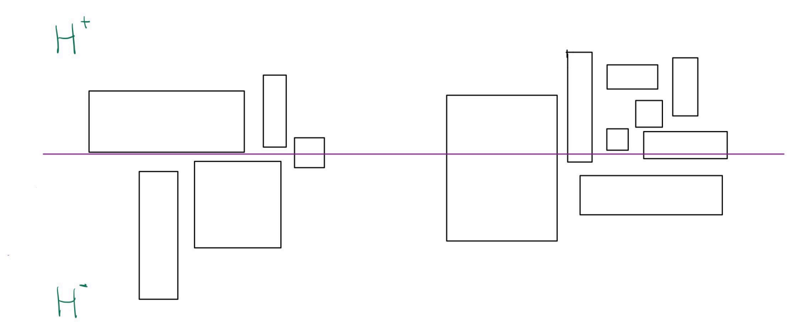
yielding $|K + L|^{\frac{1}{n}} \geq |K|^{\frac{1}{n}} + |L|^{\frac{1}{n}}$.

Step 2. Suppose K and L are finite unions of disjoint boxes. We proceed by induction on the *total* number of boxes comprising K and L. The base case with two boxes was proved in Step 1. Suppose the inequality holds for N boxes. Let $H = \theta^{\perp} + t\theta$ be a hyperplane avoiding at least one box of K, and define:

$$H^+ = \{x \in \mathbb{R}^n : \langle x, \theta \rangle > t\}, \quad H^- = \mathbb{R}^n \setminus H^+.$$

By translating L, we can assume:

$$\frac{|K \cap H^+|}{|K|} = \frac{|L \cap H^+|}{|L|} = a \in [0, 1]. \tag{3}$$



By the inductional assumption, the Brunn-Minkowski inequality holds for $K \cap H^+$ and

$L \cap H^+$, and for $K \cap H^-$ and $L \cap H^-$. Using it, together with (3), we get

$$\begin{aligned}
|K+L| &\geq |K\cap H^+ + L\cap H^+| + |K \cap H^- + L\cap H^-| \\
&\geq \left(|K\cap H^+|^{1/n} + |L\cap H^+|^{1/n}\right)^n + \left(|K\cap H^-|^{1/n} + |L\cap H^-|^{1/n}\right)^n \\
&= \left(a^{1/n}|K|^{1/n} + a^{1/n}|L|^{1/n}\right)^n + \left((1-a)^{1/n}|K|^{1/n} + (1-a)^{1/n}|L|^{1/n}\right)^n \\
&= a\left(|K|^{1/n} + |L|^{1/n}\right)^n + (1-a)\left(|K|^{1/n} + |L|^{1/n}\right)^n \\
&= \left(|K|^{1/n} + |L|^{1/n}\right)^n,
\end{aligned}$$

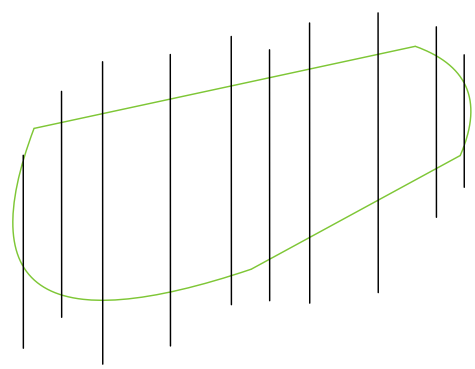
and therefore, the Brunn-Minkowski inequality holds for K and L.

Step 3. The general case follows by approximation using the definition of Borel-measurable sets as limits of unions of boxes. $\square$

1.3 Brunn's Concavity Principle

Definition 1.1 (Section function)**.** Let $\theta \in \mathbb{S}^{n-1}$. We define the *section function* $A_{\theta,K} \colon \mathbb{R} \to \mathbb{R}$ of a convex body $K \subset \mathbb{R}^n$ in the direction θ as follows:

$$A_{\theta,K}(t) = |K \cap (\theta^\perp + t\theta)|_{n-1}.$$



The following result is equivalent to the Brunn-Minkowski inequality in the case of convex sets:

Theorem 1.3 (Brunn)**.** *For any convex body K, the function $A_{\theta,K}^{\frac{1}{n-1}}(t)$ is concave on its support.*

Proof. We aim to show that for any $\lambda \in [0,1]$ and any s,t in the support of $A_{\theta,K}$,

$$\left|K \cap \left(\theta^\perp + (\lambda s + (1-\lambda)t)\theta\right)\right|^{\frac{1}{n-1}} \geq \lambda |K \cap (\theta^\perp + s\theta)|^{\frac{1}{n-1}} + (1-\lambda)|K \cap (\theta^\perp + t\theta)|^{\frac{1}{n-1}}.$$

Since K is convex, for all $x, y \in K$ we have $\lambda x + (1-\lambda)y \in K$. Also,

$$\lambda(\theta^\perp + s\theta) + (1-\lambda)(\theta^\perp + t\theta) = \theta^\perp + (\lambda s + (1-\lambda)t)\theta.$$

Therefore, we get:

$$\lambda \left(K \cap (\theta^\perp + s\theta)\right) + (1-\lambda) \left(K \cap (\theta^\perp + t\theta)\right) \subset K \cap \left(\theta^\perp + (\lambda s + (1-\lambda)t)\theta\right). \tag{4}$$

We conclude:

$$\begin{aligned}\left|K \cap \left(\theta^\perp + (\lambda s + (1-\lambda)t)\theta\right)\right|^{\frac{1}{n-1}} &\geq \left|\lambda \left(K \cap (\theta^\perp + s\theta)\right) + (1-\lambda) \left(K \cap (\theta^\perp + t\theta)\right)\right|^{\frac{1}{n-1}} \\ &\geq \lambda |K \cap (\theta^\perp + s\theta)|^{\frac{1}{n-1}} + (1-\lambda)|K \cap (\theta^\perp + t\theta)|^{\frac{1}{n-1}},\end{aligned}$$

where the last inequality follows from the Brunn–Minkowski inequality in $\mathbb{R}^{n-1}$. □

Remark 1.2. *For any subspace H of dimension k, the function $F\colon H^\perp \to \mathbb{R}$ given by $F(y) = |K \cap (H+y)|^{\frac{1}{k}}$ is concave on its support. The proof is the same and is left as a homework.*

1.4 Log-Concave Functions and Measures, Borell's Theorem, and the Prékopa–Leindler Inequality

We start by formulating two seemingly unrelated definitions.

Definition 1.2 (Log-concave function)**.** A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is log-concave if $\log f$ is concave:

$$\log f(\lambda x + (1-\lambda)y) \geq \lambda \log f(x) + (1-\lambda) \log f(y)$$

for all $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$. Equivalently,

$$f(\lambda x + (1-\lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda}.$$

In other words, $f(x) = e^{-V(x)}$ where V is convex. Note that if a function is log-concave, then its support is necessarily a convex set. Also, if f and g are log-concave, then so is fg. Examples of log-concave functions include:

- $f(x) = 1_K(x)$, where K is a convex set;
- $f(x) = e^{-\frac{x^2}{2}}$;
- $f(x) = e^{-\|x\|_M^q} \cdot 1_K(x)$ for some convex sets M and K.

Definition 1.3 (Log-concave measure)**.** A measure μ on $\mathbb{R}^n$ is log-concave if $\operatorname{supp} \mu$ has nonempty interior and for all Borel-measurable sets K, L and all $0 \leq \lambda \leq 1$,

$$\mu(\lambda K + (1-\lambda)L) \geq \mu(K)^\lambda \mu(L)^{1-\lambda}.$$

It turns out, these notions are very closely related:

Theorem 1.4 (Borell)**.** *A measure μ on $\mathbb{R}^n$ is log-concave if and only if it has a density f with respect to Lebesgue measure (possibly with respect to Lebesgue measure on some affine subspace), and f is a log-concave function.*

Remark 1.3. *Borell's theorem generalizes the Brunn–Minkowski inequality: the density of Lebesgue measure is 1, which is a log-concave function, and hence the theorem implies that Lebesgue measure is log-concave — which is equivalent to the Brunn–Minkowski inequality.*

One of the standard proofs of Theorem 1.4 uses the celebrated:

Theorem 1.5 (Prékopa–Leindler inequality (1970), the functional version of Brunn–Minkowski)**.** *Fix $\lambda \in [0,1]$. Let $f, g, h \in L^1(\mathbb{R}^n)$. Suppose for all $x, y \in \mathbb{R}^n$,*

$$h(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)g(y).$$

Then,

$$\int e^{-h} \geq \left(\int e^{-f}\right)^{\lambda} \left(\int e^{-g}\right)^{1-\lambda}.$$

Remark 1.4. *Equivalently, letting $H = e^{-h}$, $F = e^{-f}$, and $G = e^{-g}$, if*

$$H(\lambda x + (1-\lambda)y) \geq F(x)^{\lambda} G(y)^{1-\lambda},$$

then

$$\int H \geq \left(\int F\right)^{\lambda} \left(\int G\right)^{1-\lambda}.$$

Derivation of Borell's Theorem (Theorem 1.4) from the Prekopa-Leindler inequality (Theorem 1.5).

The forward direction is left as a homework. The key is the backward direction: if f is a log-concave function, then $d\mu(x) = f(x)dx$ is a log-concave measure. Indeed, let $f(x)$ be a log-concave function, let K and L be Borel-measurable sets, and let $\lambda \in [0,1]$. Define:

$$H(z) = f(z)1_{\lambda K + (1-\lambda)L}(z), \quad F(x) = f(x)1_K(x), \quad G(y) = f(y)1_L(y).$$

Then, by log-concavity of f,

$$H(\lambda x + (1-\lambda)y) \geq F(x)^{\lambda} G(y)^{1-\lambda},$$

and the Prékopa–Leindler inequality then gives:

$$\int H \geq \left(\int F\right)^{\lambda} \left(\int G\right)^{1-\lambda},$$

which amounts to

$$\mu(\lambda K + (1-\lambda)L) \geq \mu(K)^{\lambda} \mu(L)^{1-\lambda}. \quad \square$$

Therefore, we are left with the task of proving the Prekopa-Leindler inequality.

Proof of the Prékopa–Leindler Inequality

Recall the "layer-cake formula": let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a non-negative continuous function. Then, for any measure μ on $\mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} f(x)\,d\mu(x) = \int_0^\infty \mu(\{x \in \mathbb{R}^n : f(x) > t\})\,dt. \tag{5}$$

This can be proved via Fubini's theorem in $\mathbb{R}^{n+1}$ by computing the measure of the subgraph of f, i.e., the set $\{(x,t) \in \mathbb{R}^{n+1} : t \le f(x)\}$, in two different ways.

We now prove the Prékopa–Leindler inequality, using Remark 1.4, by induction on the dimension.

Step 1: $n = 1$**.** Using the layer-cake formula, we get:

$$\int_{\mathbb{R}} H(t)\,dt = \int_0^\infty |\{t \in \mathbb{R} : H(t) > s\}|\,ds.$$

Note that

$$\{H > s\} \supseteq \lambda\{F > s\} + (1-\lambda)\{G > s\}. \tag{6}$$

Using (6) and the one-dimensional Brunn–Minkowski inequality, we get:

$$\begin{aligned}
\int_{\mathbb{R}} H(t)\,dt &= \int_0^\infty |\{t \in \mathbb{R} : H(t) > s\}|\,ds \\
&\ge \int_0^\infty |\lambda\{F > s\} + (1-\lambda)\{G > s\}|\,ds \\
&\ge \lambda \int_0^\infty |\{F > s\}|\,ds + (1-\lambda)\int_0^\infty |\{G > s\}|\,ds \\
&= \lambda \int_{\mathbb{R}} F(t)\,dt + (1-\lambda)\int_{\mathbb{R}} G(t)\,dt \\
&\ge \left(\int_{\mathbb{R}} F(t)\,dt\right)^{\lambda} \left(\int_{\mathbb{R}} G(t)\,dt\right)^{1-\lambda}.
\end{aligned}$$

Step 2: Induction. Assume the claim holds for dimension $n-1$. Define:

$$H_n(x_n) := \int_{\mathbb{R}^{n-1}} H(x_1, \ldots, x_n)\,dx_1 \cdots dx_{n-1},$$

and define F_n and G_n similarly. Then $F_n, G_n, H_n \in L^1(\mathbb{R})$ by Fubini's theorem. For fixed $x_n, y_n \in \mathbb{R}$ and $\bar{x}, \bar{y} \in \mathbb{R}^{n-1}$, the assumption of Theorem 1.5 amounts to the inequality

$$H(\lambda(\overline{x}, x_n) + (1-\lambda)(\overline{y}, y_n)) \ge F(\overline{x}, x_n)^{\lambda} G(\overline{y}, y_n)^{1-\lambda}.$$

Viewing this as a function of $\overline{x}$ and $\overline{y}$, the induction hypothesis yields:

$$H_n(\lambda x_n + (1-\lambda)y_n) \ge F_n(x_n)^{\lambda} G_n(y_n)^{1-\lambda}.$$

Then, apply the one-dimensional case which was verified in step 1:

$$\int_{\mathbb{R}} H_n(t)\,dt \geq \left(\int_{\mathbb{R}} F_n(t)\,dt\right)^{\lambda} \left(\int_{\mathbb{R}} G_n(t)\,dt\right)^{1-\lambda}.$$

Finally, by Fubini's theorem, this gives the full result. □

1.5 Borell–Brascamp–Lieb Inequality

Definition 1.4. We say a function F on $\mathbb{R}^n$ is *p-concave* if F^p is concave.

It turns out that the Prékopa–Leindler inequality is a member of a more general family of inequalities:

Theorem 1.6 (Borell–Brascamp–Lieb)**.** *Suppose $p \in (-1/n, \infty)$, consider functions $f, g, h \geq 0$ on $\mathbb{R}^n$, and fix $\lambda \in [0, 1]$.*

- *If $p \geq 0$ and*
$$h(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)g(y),$$
then
$$\left(\int h^{1/p}\right)^{\frac{p}{np+1}} \geq \lambda \left(\int f^{1/p}\right)^{\frac{p}{np+1}} + (1-\lambda)\left(\int g^{1/p}\right)^{\frac{p}{np+1}}.$$
- *If $p < 0$ and*
$$h(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)g(y),$$
then
$$\left(\int h^{1/p}\right)^{\frac{p}{np+1}} \leq \lambda \left(\int f^{1/p}\right)^{\frac{p}{np+1}} + (1-\lambda)\left(\int g^{1/p}\right)^{\frac{p}{np+1}}.$$

Note that the limiting case $p = 0$ corresponds to the Prékopa–Leindler inequality.

2 From concavity principles to isoperimetry via linearizations

In what follows, we will frequently exploit the following simple idea. Suppose that $\mathcal{F}$ is a functional on some reasonable class of functions, and that $\mathcal{F}$ is concave:

$$\mathcal{F}((1-t)f + tg) \geq (1-t)\mathcal{F}(f) + t\mathcal{F}(g).$$

Then, for fixed f and g, define the univariate function

$$\alpha(t) = \mathcal{F}((1-t)f + tg) - (1-t)\mathcal{F}(f) - t\mathcal{F}(g),$$

and observe the following:

1. $\alpha'(0) \geq 0$ if $\alpha'(0)$ exists.

   Indeed, concavity of $\mathcal{F}$ implies that $\alpha(t) \geq 0$ on $[0,1]$. Also, $\alpha(0) = 0$ by definition. The conclusion follows assuming $\alpha'(t)$ exists on $[0,\epsilon]$ for some $\epsilon > 0$.

2. $\alpha''(0) \leq 0$ if $\alpha''(0)$ exists.

   This follows directly from the fact that $\mathcal{F}$ is concave, and thus $\frac{d^2}{dt^2}\mathcal{F}(f+tg) \leq 0$.

3. If $\mathcal{F}(f) \leq \mathcal{F}(f_0)$ for all f in the appropriate class, then

$$\frac{d}{d\epsilon}\mathcal{F}(f_0+\epsilon f)\bigg|_{\epsilon=0} = 0, \quad \text{and} \quad \frac{d^2}{d\epsilon^2}\mathcal{F}(f_0+\epsilon f)\bigg|_{\epsilon=0} \leq 0,$$

   provided all the derivatives are well-defined.

These observations give us a method for discovering new inequalities from existing concavity principles.

Our first goal is to understand in what sense the Prékopa–Leindler inequality can be interpreted as a concavity principle. The hypothesis of Prékopa–Leindler is that

$$h((1-t)x+ty) \leq (1-t)f(x)+tg(y).$$

What is the best possible h satisfying this inequality?

Definition 2.1 (Infimal convolution)**.** Given functions $f,g\colon \mathbb{R}^n \to \mathbb{R}$ and $t \in [0,1]$, we define the *infimal convolution*

$$f\Box_t g(z) = \inf_{(1-t)x+ty=z} \left\{(1-t)f(x)+tg(y)\right\}.$$

We also write

$$f\Box g(z) = \inf_{x+y=z} \left\{f(x)+g(y)\right\}.$$

Note that $f\Box_t g = h$ satisfies the condition of the Prékopa–Leindler inequality, and therefore:

$$\int e^{-f\Box_t g} \geq \left(\int e^{-f}\right)^{1-t} \left(\int e^{-g}\right)^{t}.$$

We also observe that $f\Box_0 g = f$ and $f\Box_1 g = g$, so in a certain sense the infimal convolution interpolates between f and g. We want to find a transform $\mathcal{T}$ which *linearizes* the infimal convolution, i.e.

$$\mathcal{T}((1-t)f+tg) = f\Box_t g$$

Then the Prekopa-Leindler inequality would amount to concavity of $\log\int e^{-\mathcal{T}(f)}$ on the appropriate linear space of functions.

Example 2.1. *When K, L are convex bodies, consider the infimal convolution of their convex indicator functions:*

$$\mathbb{1}_K^\infty \Box \mathbb{1}_L^\infty(z) = \inf_{x+y=z} \left\{\mathbb{1}_K^\infty(x)+\mathbb{1}_L^\infty(y)\right\} = \mathbb{1}_{K+L}^\infty(z).$$

Recall that $h_{K+L} = h_K + h_L$, so if we would like $\mathcal{T}$ such that $\mathcal{T}(f\Box g) = \mathcal{F}(f)+\mathcal{F}(g)$, then a good hint is that we want $\mathcal{T}\colon \mathbb{1}_K^\infty \mapsto h_K$.

2.1 Legendre Transform

We start by defining the Legendre transform of a function.

Definition 2.2. For $f : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, we define

$$f^*(x) = \sup_{y \in \mathbb{R}^n} \{\langle x, y\rangle - f(y)\}.$$

Example 2.2. *1. For a convex body K,*

$$\begin{aligned}(\mathbb{1}_K^\infty)^*(x) &= \sup_{y\in\mathbb{R}^n}\{\langle x,y\rangle - \mathbb{1}_K^\infty(y)\}\\ &= \sup_{y\in K}\{\langle x,y\rangle\}\\ &= h_K(x).\end{aligned}$$

2.

$$\begin{aligned}|x|^*(x) &= \sup_{y\in\mathbb{R}^n}\{\langle x,y\rangle - |y|\}\\ &= \sup_{t\geq 0}\{t|x| - t\}\\ &= \begin{cases} 0 & \text{if } |x| \leq 1, \\ \infty & \text{if } |x| > 1 \end{cases} = \mathbb{1}^\infty_{B_2^n}(x).\end{aligned}$$

3. Does Legendre transform have any fixed points? When is $f^* = f$? Turns out, this happens when $f(x) = |x|^2/2$. Indeed,

$$\begin{aligned}f^*(x) &= \sup_{y\in\mathbb{R}^n}\{\langle x,y\rangle - \tfrac{1}{2}|y|^2\}\\ &= \sup_{t\geq 0}\{t|x|^2 - \tfrac{1}{2}t^2|x|^2\}.\end{aligned}$$

The function inside the supremum is quadratic in t and is maximized when $t = 1$. Thus

$$f^*(x) = \tfrac{1}{2}|x|^2.$$

4. More generally, it is left as a homework to verify that for a convex body K in $\mathbb{R}^n$,

$$\left(\frac{\|x\|_K^p}{p}\right)^* = \frac{\|x\|^q_{K^o}}{q},$$

where $K^o = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \,\forall y \in K\}$ is the so-called polar body of K, and $\frac{1}{p} + \frac{1}{q} = 1$, i.e. p and q are Hölder duals. Here the Minkowski functional of K is defined as $\|x\|_K = \inf\{t > 0 : \frac{x}{t} \in K\}$, in other words, when K is symmetric then it is the norm on $\mathbb{R}^n$ for which K is the unit ball.

5. *Consider $f(x) = C$, a constant function. Then*

$$f^*(x) = \sup_{y \in \mathbb{R}^n} \{\langle x, y\rangle - C\} = \infty.$$

This does not depend on the choice of constant C.

6. *Consider*

$$f(y) = \begin{cases} -\sqrt{1-|y|^2} & \text{if } |y| \leq 1, \\ \infty & \text{if } |y| > 1. \end{cases}$$

Then

$$\begin{aligned} f^*(x) &= \sup_{y \in \mathbb{R}^n} \{\langle x, y\rangle - f(y)\} \\ &= \sup_{|y| \leq 1} \left\{\langle x, y\rangle + \sqrt{1-|y|^2}\right\} \\ &= \sup_{t \in [0, |x|^{-1}]} \left\{t|x|^2 + \sqrt{1-t^2|x|^2}\right\}. \end{aligned}$$

Optimizing by hand:

$$\frac{d}{dt}\left(t|x|^2 + \sqrt{1-t^2|x|^2}\right) = |x|^2 - \frac{t|x|^2}{\sqrt{1-t^2|x|^2}} = 0,$$

which gives

$$\frac{t}{\sqrt{1-t^2|x|^2}} = 1 \quad \Rightarrow \quad t = \frac{1}{\sqrt{1+|x|^2}}.$$

Plugging this back in:

$$\begin{aligned} f^*(x) &= \frac{|x|^2}{\sqrt{1+|x|^2}} + \sqrt{1 - \frac{|x|^2}{1+|x|^2}} \\ &= \frac{|x|^2}{\sqrt{1+|x|^2}} + \sqrt{\frac{1}{1+|x|^2}} \\ &= \frac{|x|^2+1}{\sqrt{1+|x|^2}} = \sqrt{1+|x|^2}, \end{aligned}$$

which is the upper branch of a hyperbola with asymptotes $y = \pm x$.

Lemma 2.1 (Properties of the Legendre transform)**.** *Let $\phi : \mathbb{R}^n \to \mathbb{R}$. Then:*

1. *ϕ^* is convex (as it is a supremum of affine functions).*

2. *If ϕ is convex, then $(\phi^*)^* = \phi$.*

3. *For any* $a \in \mathbb{R}$, $(\phi + a)^*(x) = \phi^*(x) - a$.

4. *If* $f(x) \le g(x)$, *then* $f^*(x) \ge g^*(x)$.

5. $(af)^*(x) = af^*(\frac{x}{a})$ *for* $a > 0$.

Proof. Home work! ☐

Next, we are ready to prove the property which we were looking for in the first place.

Proposition 2.1 (Legendre transform linearizes infimal convolution)**.** *For convex functions* f, g,

$$(f \Box g)^* = f^* + g^*.$$

Proof. This is a direct calculation:

$$\begin{aligned}
(f \Box g)^*(x) &= \sup_{y \in \mathbb{R}^n} \left\{ \langle x, y \rangle - \inf_{a+b=y} [f(a) + g(b)] \right\} \\
&= \sup_{a,b \in \mathbb{R}^n} \{ \langle x, a+b \rangle - f(a) - g(b) \} \\
&= \sup_{a \in \mathbb{R}^n} \{ \langle x, a \rangle - f(a) \} + \sup_{b \in \mathbb{R}^n} \{ \langle x, b \rangle - g(b) \} \\
&= f^*(x) + g^*(x).
\end{aligned}$$

☐

Proposition 2.2 (Legendre transform of smooth functions)**.** *Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be strictly convex and* C^2*. Then:*

1. $V(x) + V^*(\nabla V(x)) = \langle x, \nabla V(x) \rangle$,

2. $\nabla V(\nabla V^*(x)) = x$, *i.e.*, $\nabla V \circ \nabla V^* = Id$,

3. $\nabla^2 V^*(\nabla V(x)) = [\nabla^2 V(x)]^{-1}$.

Proof. 1. From the definition of V^*, at the optimal point $y = \nabla V(x)$, we have

$$V^*(\nabla V(x)) = \langle x, \nabla V(x) \rangle - V(x),$$

which rearranges to give the identity.

2. Taking gradients of both sides in (1), we get

$$\nabla^2 V(x) \cdot \nabla V^*(\nabla V(x)) = x,$$

hence $\nabla V^*(\nabla V(x)) = x$.

3. Differentiating again yields the identity for the Hessians – we leave the details as an exercise.

☐

Remark 2.1. *Relation (2) in Proposition 2.2 implies that* $V^{**} = V$, *i.e., the involution property of the Legendre transform, under the assumptions that* V *is* C^2 *and finite everywhere.*

2.2 Generalized Log-Sobolev Inequality

Recall that the Prékopa-Leindler inequality can be written as

$$\int e^{-f\Box_t g} \geq \left(\int e^{-f}\right)^t \left(\int e^{-g}\right)^{1-t}.$$

By replacing the functions with their Legendre transforms, and using the fact that

$$(tf + (1-t)g)^* = f^* \Box_t g^*,$$

we obtain the following inequality (for which, in fact, convexity is not needed):

$$\int e^{-(tf+(1-t)g)^*} \geq \left(\int e^{-f^*}\right)^t \left(\int e^{-g^*}\right)^{1-t}.$$

This convenient formulation was noted by Cordero-Erausquin and Klartag [13]. In other words, $\log \int e^{-f^*}$ *is a concave functional on the space of reasonable functions (for which the corresponding integrals exist).*

Remark 2.2. *As per [13], note also the dual fact: the functional* $\log \int e^{-f}$ *is convex by Hölder's inequality.*

Now, let

$$\alpha(t) = \log \int e^{-((1-t)f+tg)^*} - (1-t)\log \int e^{-f^*} - t \log \int e^{-g^*}.$$

By the Prekopa-Leindler inequality, $\alpha(t) \geq 0$, and note also that $\alpha(0) = 0$, and from these two facts we conclude that $\alpha'(0) \geq 0$. We will compute this derivative explicitly, and for this, we need:

Lemma 2.2. *Let $V_t(x)$ be a family of functions on $\mathbb{R}^n$ for $t \in [0,1]$ such that $V_t \in C^2(\mathbb{R}^n, \mathbb{R})$ and $V_t(x)$ is convex for each t. Then*

$$\begin{aligned}
\frac{d}{dt} V_t^*(x) &= -\dot{V}_t(\nabla V_t^*(x)); \\
\frac{d^2}{dt^2} V_t^*(x) &= -\ddot{V}_t(\nabla V_t^*(x)) + \langle (\nabla^2 V_t(x))^{-1} \nabla [\dot{V}_t|_{\nabla V_t^*(x)}], \nabla [\dot{V}_t|_{\nabla V_t^*(x)}] \rangle.
\end{aligned}$$

Proof. We will only show the first identity, and the second one is left as a home work (Question 4.12). Recall the following duality formula for the Legendre transform:

$$V_t(x) + V_t^*(\nabla V_t) = \langle \nabla V_t, x \rangle.$$

Differentiating both sides with respect to t, we get

$$\dot{V}_t(x) + \frac{d}{dt} V_t^*(\nabla V_t) + \langle \nabla V_t^*(\nabla V_t), \nabla \dot{V}_t \rangle = \langle \nabla \dot{V}_t, x \rangle.$$

Now using $\nabla V_t^* \circ \nabla V_t = x$, the above identity becomes

$$\dot{V}_t(x) + \frac{d}{dt} V_t^*(\nabla V_t) + \langle x, \nabla \dot{V}_t \rangle = \langle \nabla \dot{V}_t, x \rangle,$$

which yields

$$\frac{d}{dt} V_t^*(\nabla V_t) = -\dot{V}_t(x).$$

It remains to set $y = \nabla V_t$ and use again $\nabla V_t^* \circ \nabla V_t = x$ to complete the proof. $\square$

We are ready to prove an inequality which can be seen as a version of Minkowski's first inequality:

Theorem 2.1 (Minkowski's first inequality for functions)**.** *Suppose F, G are convex and $\int e^{-F} = \int e^{-G}$. Then*

$$\int G^*(\nabla F) e^{-F} \geq \int F^*(\nabla F) e^{-F},$$

and the left-hand side is minimized when $G = F$.

Proof. Recall that

$$\alpha(t) = \log \int e^{-((1-t)f+tg)^*} - (1-t) \log \int e^{-f^*} - t \log \int e^{-g^*},$$

and by Prekopa-Leindler inequality, $\alpha(t) \geq 0$, and since $\alpha(0) = 0$, we deduce that $\alpha'(0) \geq 0$. Let us now compute:

$$\alpha'(0) = \frac{1}{\int e^{-f^*}} \cdot \int -e^{-f^*} \cdot \frac{d}{dt}((1-t)f + tg)^* \Big|_{t=0} + \log \frac{\int e^{-f^*}}{\int e^{-g^*}} \tag{7}$$

$$= \frac{1}{\int e^{-f^*}} \cdot \int -e^{-f^*} \cdot (f - g)(\nabla f^*) + \log \frac{\int e^{-f^*}}{\int e^{-g^*}}. \tag{8}$$

When f and g are convex, let $F = f^*$ and $G = g^*$. The fact that $\alpha'(0) \geq 0$ amounts to the inequality:

$$\int -e^{-F} \cdot (F^* - G^*)(\nabla F) + \int e^{-F} \cdot \log \frac{\int e^{-F}}{\int e^{-G}} \geq 0. \tag{9}$$

It remains to consider the partial case $\int e^{-F} = \int e^{-G}$. $\square$

Remark 2.3. *Note that Theorem 2.11 implies the classical Minkowski's first inequality:*

$$V_1(K, L) \geq |K|^{\frac{n-1}{n}} |L|^{\frac{1}{n}},$$

where the so-called mixed volume $V_1(K, L) = \frac{1}{n}\frac{d}{dt}|K + tL|$. Indeed, it suffices to show this inequality in the case when $|K| = |L|$, and the rest follows by homogeneity; and the case $|K| = |L|$ follows if we plug $F = 1_K^\infty$ and $G = 1_L^\infty$ (we leave the details to the curious reader). This is not surprising, in a way: the Prekopa-Leindler inequality is the functional form of the Brunn-Minkowski inequality, and Minkowksi's first inequality is traditionally deduced from the Brunn-Minkowski in the same way as Theorem 2.11 from Prekopa-Leindler. This should give an intuition about the isoperimetric nature of Theorem 2.11.

Continuing with (9), we can use part 1 from the Proposition 2.2:

$$\int F^*(\nabla F)e^{-F} = \int (\langle \nabla F, x\rangle - F(x))e^{-F}.$$

Note that $\int \langle \nabla F(x), x\rangle e^{-F} = -\int \langle \nabla e^{-F}, x\rangle$, so we use integration by parts:

$$\int \langle \nabla F(x), x\rangle e^{-F} = -\int \langle \nabla e^{-F}, x\rangle = \int e^{-F} \cdot \Delta \frac{x^2}{2} = n \int e^{-F}.$$

Using the above, we note that the following inequality is equivalent to (9):

Theorem 2.2 (Generalized Log-Sobolev Inequality)**.** *If F, G are convex functions, then*

$$\int G^*(\nabla F)e^{-F} \geq n \int e^{-F} - \int F e^{-F} + \int e^{-F} \log \frac{\int e^{-F}}{\int e^{-G}}.$$

Note that the equality holds if $F = G$.

Corollary 2.1. *If F, G are convex functions and $\int e^{-F} = \int e^{-G}$, then*

$$\int G^*(\nabla F)e^{-F} \geq n \int e^{-F} - \int F e^{-F}.$$

Remark 2.4. *The derivation of the classical Log-Sobolev inequality (which we will discuss shortly) using linearization of the Prékopa-Leindler inequality was first done by Bobkov and Ledoux.*

2.3 Reformulations and Notable Partial Cases of the Generalized Log-Sobolev Inequality

Now let us state a reformulation of Theorem 2.2. Consider $\phi = e^{-F}$ for some convex F. Then $F = -\log \phi$ and $\nabla F = -\nabla \phi / \phi$. Assume G is convex and $\int e^{-G} = 1$, then Theorem 2.2 can be rewritten as

$$\int G^* \left(-\frac{\nabla \phi}{\phi}\right) \phi \geq n \int \phi + \int \phi \log \phi - \left(\int \phi\right) \log \left(\int \phi\right).$$

Definition 2.3 (Entropy)**.** If $d\mu$ is a measure on $\mathbb{R}^n$, then the entropy of a function ϕ with respect to the measure μ is defined as

$$\mathrm{Ent}_\mu(\phi) := \int \phi \log \phi \, d\mu - \left(\int \phi \, d\mu\right) \log \left(\int \phi \, d\mu\right).$$

When μ is Lebesgue, we write $\mathrm{Ent}(\phi)$ for simplicity.

Remark 2.5. *By Jensen's inequality, using the convexity of $t \log t$, we get $Ent_\mu(\phi) \geq 0$ for any probability measure μ.*

Theorem 2.3 (Reformulation of the Generalized Log-Sobolev Inequality)**.** *For any log-concave function ϕ and convex function G with $\int e^{-G} = 1$, we have*

$$\int G^*\left(-\frac{\nabla \phi}{\phi}\right)\phi \geq n\int \phi + Ent(\phi).$$

We derive, by plugging in $G^*(x) = |x| - \log|B_2^n|$:

Corollary 2.2 (L_1-Sobolev inequality, Bobkov–Ledoux, 2000)**.** *For a log-concave function ϕ,*

$$Ent(\phi) + C_n \int \phi \leq \int |\nabla \phi|,$$

where $C_n = n + \log|B_2^n|$.

At last, by plugging in $G^*(x) = \frac{|x|^2}{2} - n\log\sqrt{2\pi}$, we obtain the classical Log-Sobolev inequality:

Corollary 2.3 (Classical Lebesgue Log-Sobolev inequality, first form)**.**

$$\textit{(1st form)} \quad Ent(\phi) + n\log(\sqrt{2\pi}e)\int \phi \leq \frac{1}{2}\int \frac{|\nabla\phi|^2}{\phi}.$$

Note that $\int \frac{|\nabla \phi|^2}{\phi}$ is called the (Lebesgue) Fisher information. By substituting $\phi = f^2$, one can also write:

Corollary 2.4 (Classical Lebesgue Log-Sobolev inequality, second form)**.**

$$\textit{(2nd form)} \quad Ent(f^2) + n\log(\sqrt{2\pi}e)\int f^2 \leq 2\int |\nabla f|^2.$$

Remark 2.6. *In Corollaries 2.3 and 2.4, one does not need to assume that ϕ is log-concave—see homework.*

We also note, by plugging in $G^* = |x|^p + C(n,p)$:

Corollary 2.5 (L_p-Sobolev inequality)**.**

$$Ent(f^p) + C_{n,p}\int f^p \leq p^{p-1}\int |\nabla f|^p,$$

where $C_{n,p} = n - \log\left(|S^{n-1}| \cdot \int_0^\infty t^{n-1}e^{-t^p/q}dt\right)$, with $\frac{1}{p} + \frac{1}{q} = 1$.

Finally, we state:

Theorem 2.4 (Gaussian Log-Sobolev inequality)**.** *Let $d\gamma$ be the standard Gaussian measure in $\mathbb{R}^n$, and let $g \in W^{1,2}(d\gamma)$ be log-concave, then*

$$Ent_\gamma(g^2) \leq 2 \int |\nabla g|^2 d\gamma.$$

Remark 2.7. *Log-concavity assumption on g is not needed—see homework.*

Remark 2.8. *Theorem 2.4 is equivalent to Corollary 2.4 by choosing $g = (2\pi)^{\frac{n}{4}} e^{\frac{|x|^2}{4}} f$. See homework.*

Remark 2.9. *The function $g = 1$ gives the equality case in Theorem 2.4.*

Remark 2.10. *Theorem 2.4 (or Corollary 2.3, since they are equivalent) implies the Lebesgue Sobolev inequality:*

$$n|B_2^n|^{\frac{1}{n}} \left(\int_{\mathbb{R}^n} |f|^{\frac{n-1}{n}} dx \right)^{\frac{n}{n-1}} \leq \int_{\mathbb{R}^n} |\nabla f| dx,$$

which holds for all smooth f such that the integral converges.

Remark 2.11. *The inequality also holds on the sphere S^{n-1}, which actually implies Theorem 2.4:*

$$\int_{S^{n-1}} f^2 \log f^2 - \left(\int_{S^{n-1}} f^2 \right) \log \left(\int_{S^{n-1}} f^2 \right) \leq 2 \int_{S^{n-1}} |\nabla_{S^{n-1}} f|^2.$$

Lastly, we mention the following fact (whose proof is left as a homework problem):

Theorem 2.5 (Generalized Log-Sobolev inequality for log-concave measures)**.** *Let $d\mu = e^{-V} dx$ be a log-concave measure and F, G be convex functions such that $\int e^{-G} d\mu = 1$. Then*

$$Ent_\mu(e^{-F}) + n \int e^{-F} d\mu - \int \langle \nabla V, x \rangle e^{-F} d\mu \leq \int G^*(\nabla F) e^{-F} d\mu.$$

Proof. Homework! □

2.4 The p-Beckner Inequality

We mention, without proof:

Theorem 2.6 (p-Beckner inequality)**.** *For $f \in W^{1,2}(\mathbb{R}^n, \gamma)$ and $p \in [1, 2)$,*

$$\int f^2 d\gamma - \left(\int |f|^p d\gamma \right)^{\frac{2}{p}} \leq (2 - p) \int |\nabla f|^2 d\gamma.$$

This result implies the so-called Gaussian Poincaré inequality when $p = 1$:

$$\int f^2 d\gamma - \left(\int f d\gamma\right)^2 \leq \int |\nabla f|^2 d\gamma.$$

We will formally prove this fact soon. Also, Beckner's inequality implies the Gaussian Log-Sobolev inequality: one can obtain

$$\mathrm{Ent}_\gamma(f) \leq 2 \int |\nabla f|^2 d\gamma$$

by letting $p \to 2$ in Theorem 2.6 and taking the derivative; see homework.

Remark 2.12. *The p-Beckner inequality is stronger when p is bigger. In other words, the Gaussian Log-Sobolev inequality is the strongest inequality in the family of all p-Beckner inequalities.*

Remark 2.13. *The analogous inequality is also known to hold on the sphere S^{n-1}.*

Below we fix a log-concave probability measure $d\mu = e^{-G}dx$ on $\mathbb{R}^n$, i.e., we assume that G is a smooth convex function and $\int e^{-G} = 1$.

Definition 2.4 (Variance)**.** The variance of a function ϕ with respect to μ is defined as

$$\mathrm{Var}_\mu(\phi) := \int \phi^2 \, d\mu - \left(\int \phi \, d\mu\right)^2.$$

Note that the variance of a constant function is zero. Also, variance is invariant under adding constants to ϕ. In a sense, variance measures "how far" ϕ is from a constant function.

2.5 A few words about the Laplace operator with respect to log-concave measures

Definition 2.5 (Laplace operator associated to μ)**.** Let μ be a log-concave measure with density e^{-V} on $\mathbb{R}^n$. Here V is a convex function. For a "reasonable" function u,

$$L_\mu u := \Delta u - \langle \nabla V, \nabla u \rangle.$$

Example 2.3. *1. If μ is Lebesgue measure, then $L_\mu u = \Delta u$.*

2. If μ is Gaussian, then $L_\mu u = \Delta u - \langle x, \nabla u \rangle$, and it is called is called the Ornstein–Uhlenbeck operator.

Lemma 2.3 (Integration by parts)**.** *If $u, v \in C^2(\mathbb{R}^n)$ and the integrals converge, then*

$$\int u L_\mu v \, d\mu = -\int \langle \nabla u, \nabla v \rangle \, d\mu.$$

Proof. Using classical integration by parts $\int f\Delta g = -\int\langle\nabla f,\nabla g\rangle$, we compute:

$$\begin{aligned}\int uL_\mu v\,d\mu &= \int(ue^{-V})\Delta v - \int u\langle\nabla V,\nabla v\rangle e^{-V}\\ &= -\int\langle\nabla(ue^{-V}),\nabla v\rangle - \int u\langle\nabla V,\nabla v\rangle e^{-V}\\ &= -\int\langle\nabla u,\nabla v\rangle e^{-V} + \int u\langle\nabla V,\nabla v\rangle e^{-V} - \int u\langle\nabla V,\nabla v\rangle e^{-V}\\ &= -\int\langle\nabla u,\nabla v\rangle\,d\mu.\end{aligned}$$

□

2.6 A short and non-standard proof sketch of integration by parts

We now give a non-standard sketch of the integration by parts formula using a change of variables argument. This works for Lebesgue measure and more generally for log-concave measures.

Lemma 2.4 (Integration by parts via change of variables). *Let $d\mu = e^{-V}dx$, with $L_\mu u = \Delta u - \langle\nabla V,\nabla u\rangle$, and let $f,g\colon\mathbb{R}^n\to\mathbb{R}$, with g smooth and bounded. Then*

$$\int f\cdot L_\mu g\,d\mu = -\int\langle\nabla f,\nabla g\rangle\,dx.$$

In particular, when $V=0$ (i.e., for Lebesgue measure),

$$\int f\cdot\Delta g\,dx = -\int\langle\nabla f,\nabla g\rangle\,dx.$$

Proof. Consider the change of variable $x = y + t\nabla g(y)$, whose Jacobian is $\det(\mathrm{Id}+t\nabla^2 g)$. Then:

$$\int f(x)\,d\mu(x) = \int f(y+t\nabla g(y))\,e^{-v(y+t\nabla g(y))}\det(\mathrm{Id}+t\nabla^2 g)\,dy.$$

Since the left-hand side is independent of t, differentiating with respect to t gives:

$$\frac{d}{dt}\int f(y+t\nabla g(y))e^{-V(y+t\nabla g(y))}\det(\mathrm{Id}+t\nabla^2 g)\,dy\bigg|_{t=0} = 0.$$

Differentiating under the integral:

$$0 = \int\left(\langle\nabla f,\nabla g\rangle e^{-V} - f\langle\nabla V,\nabla g\rangle e^{-V} + f\cdot\Delta g\,e^{-V}\right)dy.$$

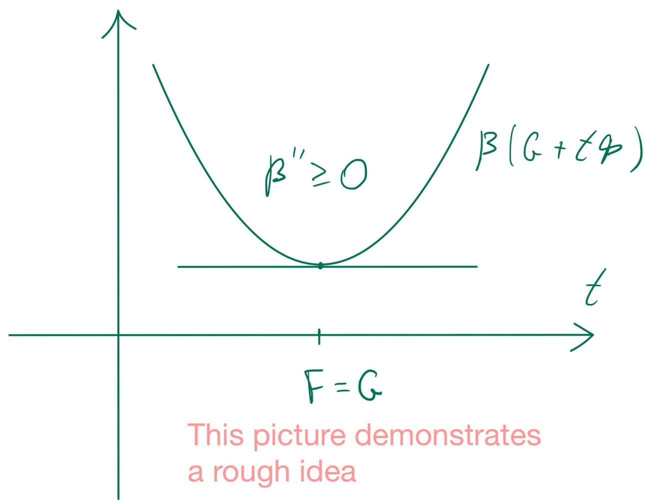
This implies:

$$\int f\cdot L_\mu g\,d\mu = -\int\langle\nabla f,\nabla g\rangle\,d\mu.$$

□

2.7 The Derivation of the Brascamp–Lieb Inequality from the Generalized Log-Sobolev Inequality

The idea: Recall that we obtained the Generalized Log-Sobolev inequality from the Prékopa–Leindler inequality by taking the first derivative near the point of minimum. Now we will continue with this approach: derive new inequalities by taking further derivatives of Prékopa–Leindler (or Generalized Log-Sobolev) around the point of maximum.



Recall the Generalized Log-Sobolev inequality: For convex functions F, G with $\int e^{-G} = 1$,

$$\int G^*(\nabla F)e^{-F} \geq n\int e^{-F} - \int Fe^{-F} - \int e^{-F}\log\int e^{-F},$$

with equality when $F = G$. To take the derivative around this point, let $F = G + t\phi$ and define

$$\beta(t) := \int G^*(\nabla G + t\nabla\phi)e^{-G-t\phi} - n\int e^{-G-t\phi} + \int(G+t\phi)e^{-G-t\phi} + \int e^{-G-t\phi}\log\int e^{-G-t\phi}.$$

We have $\beta(t) \geq 0$ and $\beta(0) = 0$. We will see that also $\beta'(0) = 0$ (as expected at a point of minimum), and this will imply that $\beta''(0) \geq 0$, which leads to a beautiful inequality called the Brascamp–Lieb inequality.

1. Write the Taylor expansion of G^* up to second order:

$$G^*(\nabla G + t\nabla\phi) = G^*(\nabla G) + t\langle\nabla G^*(\nabla G), \nabla\phi\rangle + \frac{t^2}{2}\langle\nabla^2 G^*(\nabla G)\nabla\phi, \nabla\phi\rangle + o(t^2).$$

From now on, we drop all $o(t^2)$ terms.

2. Use $e^{-\delta} = 1 - \delta + \frac{\delta^2}{2}$ to obtain

$$e^{-G-t\phi} = e^{-G}(1 - t\phi + \frac{t^2}{2}\phi^2).$$

Combining this with $\log(1+\delta) = \delta - \frac{\delta^2}{2}$ and using $\int e^{-G} = 1$, we get

$$\begin{aligned}\log\left(\int e^{-G-t\phi}\right) &= \log\left(1 - t\int \phi e^{-G} + \frac{t^2}{2}\int \phi^2 e^{-G}\right)\\ &= -t\int \phi e^{-G} + \frac{t^2}{2}\int \phi^2 e^{-G} - \frac{t^2}{2}\left(\int \phi e^{-G}\right)^2. \qquad (*)\end{aligned}$$

Using the variance notation, $(*)$ becomes

$$\log\left(\int e^{-G-t\phi}\right) = -t\int \phi\, d\mu + \frac{t^2}{2}\mathrm{Var}_\mu(\phi).$$

Now compute each term in $\beta(t)$:
First term:

$$\begin{aligned}\int G^*(\nabla G + t\nabla\phi)e^{-G-t\phi} &= \int \left(G^*(\nabla G) + t\langle \nabla G^*(\nabla G), \nabla\phi\rangle + \frac{t^2}{2}\langle \nabla^2 G^*(\nabla G)\nabla\phi, \nabla\phi\rangle\right)\\ &\qquad \times (1 - t\phi + \frac{t^2}{2}\phi^2)\, d\mu\\ &= \int G^*(\nabla G)d\mu + t\int (\langle \nabla\phi, x\rangle - \phi G^*(\nabla G))d\mu\\ &\quad + \frac{t^2}{2}\int \left(\langle (\nabla^2 G)^{-1}\nabla\phi, \nabla\phi\rangle + \phi^2 G^*(\nabla G) - 2\phi\langle\nabla\phi, x\rangle\right) d\mu.\end{aligned}$$

The remaining terms combine to:

$$\begin{aligned}&-n\int e^{-G-t\phi} + \int (G + t\phi)e^{-G-t\phi} + \int e^{-G-t\phi}\log\int e^{-G-t\phi}\\ &= -n + \int G d\mu + t(n\int \phi d\mu - \int \phi G d\mu) + \frac{t^2}{2}\left(\int (G-n)\phi^2 d\mu - \mathrm{Var}_\mu(\phi)\right).\end{aligned}$$

Putting things together:

- By the duality formula $G + G^*(\nabla G) = \langle \nabla G, x\rangle$, and $\int \langle \nabla G, x\rangle d\mu = n$, we have

$$\int G^*(\nabla G)d\mu = n - \int G d\mu.$$

  This shows $\beta(0) = 0$.

- Using integration by parts (Lemma 2.3), one shows that

$$\int (\langle \nabla\phi, x\rangle - \phi G^*(\nabla G))d\mu + n\int \phi d\mu - \int \phi G d\mu = 0,$$

  so $\beta'(0) = 0$.

- Therefore, from the second-order terms we obtain:

$$\int \big(\langle (\nabla^2 G)^{-1}\nabla\phi, \nabla\phi\rangle + \phi^2 G^*(\nabla G) - 2\phi\langle \nabla\phi, x\rangle\big)\, d\mu + \int (G-n)\phi^2 d\mu - \mathrm{Var}_\mu(\phi) \geq 0.$$

- Finally, we show that

$$\int (\phi^2 G^*(\nabla G) - 2\phi\langle \nabla\phi, x\rangle)d\mu + \int (G-n)\phi^2 d\mu = 0,$$

  using the duality formula and Lemma 2.3.

Hence we conclude:

Theorem 2.7 (Brascamp–Lieb inequality, 1976)**.** *Let G be a strictly convex function with $\int e^{-G} = 1$, and let $d\mu = e^{-G}dx$. Then for any locally Lipschitz function ϕ, we have*

$$Var_\mu(\phi) \leq \int \langle (\nabla^2 G)^{-1}\nabla\phi, \nabla\phi\rangle d\mu.$$

Remark 2.14 (Brascamp–Lieb inequality is the local form of Prékopa–Leindler!)**.** *This inequality is equivalent to:*

$$\frac{d^2}{dt^2}\log\int e^{-(f+tg)^*} \leq 0,$$

by substituting $G = f^*$ and $\phi = g(\nabla f^*)$. In fact, Brascamp–Lieb also implies Prékopa–Leindler "by integration"; see homework.

Remark 2.15. *Equality holds in the Brascamp–Lieb inequality when $\phi = \langle \nabla G, \theta\rangle$ for any $\theta \in \mathbb{R}^n$.*

Remark 2.16. *One might try to iterate the linearization idea, setting $\phi = \langle \nabla G, \theta\rangle + \epsilon f$, but this does not yield further inequalities. The Brascamp–Lieb inequality is "the end of the line"; see homework.*

Corollary 2.6 (Gaussian Poincaré inequality)**.** *If $d\mu = d\gamma$ is the Gaussian measure, then $\nabla^2 G = Id$, and the inequality becomes:*

$$Var_\gamma(\phi) \leq \int |\nabla\phi|^2 d\gamma.$$

Remark 2.17. *This implies that the first eigenvalue of L_γ is 1. The corresponding eigenfunctions are linear: $L_\gamma\langle x,\theta\rangle = -\langle x,\theta\rangle$.*

Remark 2.18. *One may deduce this inequality also from the classical Gaussian Log-Sobolev inequality using simpler calculations; see homework.*

Remark 2.19 (An important observation about Brascamp–Lieb)**.** *If $G = V + W$ and $d\nu = e^{-(V+W)}dx$, then:*

$$Var_\nu(\phi) \le \int \langle (\nabla^2 V)^{-1}\nabla\phi, \nabla\phi\rangle d\nu,$$

provided V, W are convex.

Corollary 2.7. *Let $d\mu = e^{-V}dx$ be a log-concave measure and $K \subset \mathbb{R}^n$ a convex body. Then for any sufficiently regular function ϕ,*

$$\frac{1}{\mu(K)}\int_K \phi^2 d\mu - \left(\frac{1}{\mu(K)}\int_K \phi d\mu\right)^2 \le \frac{1}{\mu(K)}\int_K \langle (\nabla^2 V)^{-1}\nabla\phi, \nabla\phi\rangle d\mu.$$

Corollary 2.8 (Extension of the Gaussian Poincaré inequality)**.** *If $d\mu = e^{-V}dx$ is a probability measure and $\nabla^2 V \ge k \cdot Id$, then*

$$Var_\mu(\phi) \le \frac{1}{k}\int |\nabla\phi|^2 d\mu.$$

2.8 Dimensional Extensions of Generalized Log-Sobolev and Brascamp–Lieb Inequalities

Recall the following corollary of the Borell–Brascamp–Lieb inequality:

Corollary 2.9. *For $p \in [-1/n, 0]$ and f, g convex, the function*

$$\left(\int_{\mathbb{R}^n} (f^* + tg^*)^{1/p}\right)^{\frac{p}{np+1}}$$

is concave in t.

Corollary 2.10 (Bolley, Gentil, Guillin [8])**.** *For $q \in (-\infty, -n]$, and a probability measure $\mathrm{d}\mu = e^{-V}\,\mathrm{d}x$ satisfying*

$$\nabla^2 V - \frac{\nabla V \otimes \nabla V}{q} \succeq 0,$$

for all locally Lipschitz g, we have

$$\int (g \cdot e^{V/q})^2\,\mathrm{d}\mu - \left(\int g \cdot e^{V/q}\,\mathrm{d}\mu\right)^2 \le \frac{-q}{-q+1}\int \left\langle \left(\nabla^2 V - \frac{\nabla v \otimes \nabla V}{q}\right)^{-1} \nabla g, \nabla g\right\rangle \mathrm{d}\mu.$$

Remark 2.20. *When $q \to -\infty$, we recover Brascamp–Lieb. When $q = -n$, we obtain*

$$\int \left(g \cdot e^{-V/n}\right)^2 \mathrm{d}\mu - \left(\int g \cdot e^{-V/n} \,\mathrm{d}\mu\right)^2 \le \frac{n}{n+1} \int \left\langle \left(\nabla^2 V + \frac{\nabla v \otimes \nabla V}{n}\right)^{-1} \nabla g, \nabla g \right\rangle \mathrm{d}\mu.$$

Theorem 2.8 (Bolley, Cordero-Erausquin, Fujita, Gentil, Guillin [9])**.** *Let h, g, w be Borel-measurable functions satisfying for all $x, y \in \mathbb{R}^n$ and $t \in [0,1]$,*

$$h((1-t)x + ty) \le (1-t)g(x) + tw(y),$$

and $\int w^{-n} = \int g^{-n} = 1$. Then

$$\int h^{1-n} \ge (1-t) \int g^{1-n} + t \int w^{1-n}.$$

Corollary 2.11 (Convex Sobolev inequality, extension from [9])**.** *Let $n \ge 2$, and let $w \colon \mathbb{R}^n \to (0,\infty)$ satisfy $\liminf_{x\to\infty} \frac{w(x)}{\|x\|^\gamma} > 0$ for some $\gamma > \frac{n}{n-1}$. For nonnegative $g \colon \mathbb{R}^n \to \mathbb{R}$ such that $\int g^{-n} = \int w^{-n} = 1$, we have*

$$\int w^*(\nabla g)\, g^{-n} \ge \frac{1}{n-1} \int w^{1-n}.$$

Remark 2.21. *Plug in*

$$w(x) = \left(1 + \frac{|x|^q}{q}\right) C_q, \quad g = f^{\frac{p}{p-n}},$$

to recover the Sobolev inequality:

$$\|f\|_{p^*} \le \frac{\|h_p\|_{p^*}}{\left(\int \|\nabla h_p\|^p\right)^{1/p}} \left(\int \|\nabla f\|^p\right)^{1/p},$$

where $h_p(x) = \left(1 + |x|^{\frac{p}{p-1}}\right)^{\frac{p-n}{p}}$ and $p^* = \frac{np}{n-p}$.

2.9 Homework

Question 2.1 (2 points)**.** *a) Show that if K and L are Borel measurable, then $K + L$ is Borel measurable.*
b) Find an example of K and L that are Lebesgue measurable but such that $K + L$ is not Lebesgue measurable.

Question 2.2 (1 point)**.** *Below, S_u stands for Steiner symmetrization with respect to $u^\perp$; K denotes a convex body in $\mathbb{R}^n$ with non-empty interior. Show that:*
a) $S_u(aK) = aS_u(K)$ for all $a > 0$;
b) If $K \subset L$, then $S_u(K) \subset S_u(L)$; conclude that $S_u(K)$ is continuous with respect to the Hausdorff metric;
c) $S_u(K) + S_u(L) \subset S_u(K + L)$.

Question 2.3 (1 point). *Recall that for a compact set $A \subset \mathbb{R}^n$, the diameter is defined as*

$$\operatorname{diam}(A) = \max_{x,y\in A} |x-y|.$$

Prove that

$$\operatorname{diam}(S_u(K)) \leq \operatorname{diam}(K).$$

Conclude the isodiametric *inequality: if the volume of a set is fixed, its diameter is minimized by a Euclidean ball.*

Question 2.4 (1 point). *Prove that the Steiner symmetrization decreases the perimeter of a convex set. Note that this gives another proof of the isoperimetric inequality for convex sets.*

Question 2.5 (1 point). *Recall that for a convex set $K \subset \mathbb{R}^n$, the in-radius is*

$$r(K) = \sup\{t > 0 : \exists y \in \mathbb{R}^n \text{ such that } y + tB_2^n \subset K\},$$

and the circumradius is

$$R(K) = \inf\{t > 0 : \exists y \in \mathbb{R}^n \text{ such that } K \subset y + tB_2^n\}.$$

a) Prove that $r(S_u(K)) \geq r(K)$.
b) Prove that $R(S_u(K)) \leq R(K)$.
Conclude that the Euclidean ball maximizes the in-radius and minimizes the circumradius among convex bodies of fixed volume.

Question 2.6 (2 points). *Prove the Urysohn inequality. Define the* mean width *of a convex body K as*

$$w(K) = \frac{2}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} h_K(\theta)\, d\theta.$$

Show that if $|K| = |B_2^n|$, then $w(K) \geq 2$.
Hint: use the Brunn–Minkowski inequality and Steiner symmetrization.

Question 2.7 (1 point). *Fix Borel measurable sets $K, L \subset \mathbb{R}^n$. Confirm the following implication (discussed in class): if for every $\lambda \in [0,1]$,*

$$|\lambda K + (1-\lambda)L| \geq |K|^\lambda |L|^{1-\lambda},$$

then it follows that

$$|\lambda K + (1-\lambda)L|^{\frac{1}{n}} \geq \lambda |K|^{\frac{1}{n}} + (1-\lambda)|L|^{\frac{1}{n}}.$$

Question 2.8 (1 point). *Show that for $a, b > 0$, one has*

$$(\lambda a^p + (1-\lambda)b^p)^{\frac{1}{p}} \to a^\lambda b^{1-\lambda} \quad \text{as } p \to 0.$$

Question 2.9 (2 points). *a) Let $p \geq -\frac{1}{n}$, and suppose functions f, g, and h on $\mathbb{R}^n$ satisfy*

$$h(\lambda x + (1-\lambda)y) \geq ((1-\lambda)f^p(x) + \lambda g^p(y))^{\frac{1}{p}}.$$

Show that

$$\int h \geq \left((1-\lambda)\left(\int f\right)^{\frac{p}{np+1}} + \lambda\left(\int g\right)^{\frac{p}{np+1}}\right)^{\frac{np+1}{p}}.$$

Hint: try a proof similar to Lyusternik's proof of the Brunn–Minkowski inequality.
b) Conclude that if a measure's density is supported on a convex set with non-empty interior and is p-concave, then the measure is $\frac{p}{np+1}$-concave.
c) Deduce that if the density of a measure μ on $\mathbb{R}^n$ is p-concave, then the density of any marginal measure $\pi_H(\mu)$ is $\frac{p}{kp+1}$-concave, where H is an $(n-k)$-dimensional subspace (generalizing Brunn's principle).

Question 2.10 (2 points). *A function f on $\mathbb{R}^n$ is called* unconditional *if it is invariant under coordinate reflections: $f(\epsilon_1 x_1, \ldots, \epsilon_n x_n) = f(x)$ for all $\epsilon_i \in \{-1, 1\}$. A set K is called unconditional if 1_K is unconditional.*

Suppose K is an unconditional convex body and V is an unconditional convex function on $\mathbb{R}^n$. Define the measure $d\mu(x) = e^{-V(x)}\,dx$. Show that $\log \mu(e^t K)$ is a concave function of $t \in \mathbb{R}$.
Hint: Pass the integration to the positive orthant $\{x \in \mathbb{R}^n : x_i \geq 0\ \forall i\}$ and use a change of variables in the Prékopa–Leindler inequality, setting $(x_1, \ldots, x_n) = (e^{t_1}, \ldots, e^{t_n})$.

Question 2.11 (1 point). *a) Prove Minkowski's first inequality:*

$$V_1(K, L) \geq |K|^{\frac{n-1}{n}} |L|^{\frac{1}{n}}$$

(similar to the isoperimetric inequality discussed in class).
b) Prove Minkowski's quadratic inequality: for convex bodies K and L in $\mathbb{R}^n$,

$$V_2(K, L) \cdot |K| \leq V_1(K, L)^2.$$

Hint: Use the Brunn–Minkowski inequality to extract information about $\frac{d^2}{dt^2}|K + tL|^{1/n}$.

Question 2.12 (1 point). *(Added at Alex's request.) Give an example of a (rough, non-convex) set K such that the limit*

$$\lim_{\epsilon \to 0} \frac{|K + \epsilon B_2^n| - |K|}{\epsilon}$$

does not exist, and

$$\liminf_{\epsilon \to 0} \frac{|K + \epsilon B_2^n| - |K|}{\epsilon} < \limsup_{\epsilon \to 0} \frac{|K + \epsilon B_2^n| - |K|}{\epsilon}.$$

Question 2.13 (1 point)**.** *Show that any convex function $V : \mathbb{R}^n \to \overline{\mathbb{R}}$ is*
a) continuous on the support of e^{-V} (i.e., on the set where V does not take the value ∞);
b) of class C^2 almost everywhere on the support of e^{-V}.

Question 2.14 (1 point)**.** *a) Suppose $V \in C^2(\mathbb{R}^n)$. Show that for all $z_1, z_2 \in \mathbb{R}^n$,*

$$V\left(\frac{z_1 + z_2}{2}\right) + \beta(z_1, z_2) = \frac{V(z_1) + V(z_2)}{2}, \tag{10}$$

where, letting $z(t) = \frac{(1-t)z_1+(1+t)z_2}{2}$, we have

$$\beta(z_1, z_2) = \frac{1}{8}\int_{-1}^{1} (1 - |t|)\langle \nabla^2 V(z(t))(z_1 - z_2), z_1 - z_2\rangle \, dt \geq 0. \tag{11}$$

b) Conclude that the convexity of a C^2-smooth function is equivalent to the non-negativity of its Hessian.

Question 2.15 (2 points)**.** *a) Show that for any pair of convex bodies K and L, the function $|K + tL|$ is a polynomial in t of degree n.*
b) Conclude that

$$|K + tL| = \sum_{k=0}^{n} \binom{n}{k} V_k(K, L)t^k.$$

This is called the Steiner polynomial.

Question 2.16 (2 points)**.** *Let K be a convex set. Define the Gauss map $\nu_K : \partial K \to \mathbb{S}^{n-1}$ by $\nu_K(x) = \{n_x\}$, where n_x is the outer normal to ∂K at x (this set is a singleton almost everywhere). Define the* surface area measure *S_K on the sphere $\mathbb{S}^{n-1}$ by*

$$S_K(\Omega) = |\nu_K^{-1}(\Omega)|_{n-1}$$

for every Borel measurable $\Omega \subset \mathbb{S}^{n-1}$. Here $|\cdot|_{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure, i.e., for $M \subset \partial K$, $|M|_{n-1} = \int_M$ in the usual sense.

a) Show that for any pair of convex bodies K and L,

$$V_1(K, L) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_L(\theta) \, dS_K(\theta).$$

In particular,

$$|K| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K(\theta) \, dS_K(\theta).$$

b) Use Minkowski's first inequality to deduce that the surface area measure determines a convex body uniquely up to translation. That is, if $dS_K = dS_L$, then $K = L + v$ for some vector v.

Question 2.17 (2 points). *Let K be a convex body. The projection of K onto the hyperplane $\theta^\perp$ for some $\theta \in \mathbb{S}^{n-1}$ is defined as*

$$K|\theta^\perp = \{x \in \theta^\perp : \exists t \in \mathbb{R} \text{ such that } x + t\theta \in K\}.$$

a) Prove the Cauchy projection formula for a symmetric convex body K:

$$|K|\theta^\perp|_{n-1} = \frac{1}{2}\int_{\mathbb{S}^{n-1}} |\langle \theta, u\rangle|\, dS_K(u).$$

b) Suppose K and L are symmetric convex bodies such that for every $\theta \in \mathbb{S}^{n-1}$,

$$|K|\theta^\perp|_{n-1} = |L|\theta^\perp|_{n-1}.$$

Conclude that $K = L + v$ for some $v \in \mathbb{R}^n$.

Question 2.18 (1 point). *Let $h \in C^2(\mathbb{R}^2)$ be the support function of a strictly convex, compact region $K \subset \mathbb{R}^2$. Show that the surface area measure f_K has a density given by*

$$f_K(u) = h(u) + \ddot{h}(u)$$

for all $u \in \mathbb{S}^1$. Note: $h + \ddot{h}$ is invariant under translations of K.

Question 2.19 (10 points). *Prove (perhaps using elementary Harmonic Analysis?) that for every pair of π-*periodic *infinitely smooth functions ψ and h on $[-\pi, \pi]$, such that $h + \ddot{h} > 0$ and $h > 0$, one has*

$$\left(\int_{-\pi}^{\pi} (h^2 - \dot{h}^2)\, du\right)\left(\int_{-\pi}^{\pi} \left(\psi^2 - \dot{\psi}^2 + \psi^2 \frac{h + \ddot{h}}{h}\right) du\right) \le 2\left(\int_{-\pi}^{\pi} (h\psi - \dot{h}\dot{\psi})\, du\right)^2. \tag{12}$$

(Note: the assumption is π-periodic rather than 2π-periodic. I can provide an explanation or motivation upon request.)

Question 2.20 (2 points). *Prove the Rogers-Shephard inequality. For a convex body $K \subset \mathbb{R}^n$, define the* difference body

$$K - K = \{x - y : x, y \in K\}.$$

Show that

$$|K - K| \le \binom{2n}{n} |K|.$$

Hint: Use the Brunn–Minkowski inequality to show that $|K \cap (x + K)|^{1/n}$ is a concave function supported on $K - K$, and can be estimated from below by $1 - \rho_{K-K}(x)$. Using this estimate (among others), show that

$$|K|^2 = \int_{K-K} |K \cap (x + K)|\, dx \ge \binom{2n}{n}^{-1} |K| \cdot |K - K|.$$

Question 2.21 (2 points)**.** *Prove the Grünbaum inequality: Let K be a convex body whose barycenter is at the origin (i.e., $\int_K x\,dx = 0$). Show that for any $\theta \in \mathbb{S}^{n-1}$, one has*

$$|\{x \in K : \langle x, \theta\rangle \geq 0\}| \geq \left(\frac{n}{n+1}\right)^n |K| \geq \frac{|K|}{e}.$$

Question 2.22 (3 points)**.** *Prove Busemann's theorem: Given $x \in \mathbb{R}^n \setminus \{0\}$, the function $\frac{|x|}{|x^{\perp}\cap K|}$ is convex on $\mathbb{R}^n$. Conclude that it defines a norm. The unit ball of this norm is called the* intersection body *of K.*

Question 2.23. *Derive the Santalo formula for the area of a convex region in $\mathbb{R}^2$:*

$$|K| = \frac{1}{2}\int_{-\pi}^{\pi}\left(h^2 - \dot{h}^2\right)\,dt,$$

where h is the support function of K.
Hint: Use Questions 2.18 and 2.16.

Question 2.24 (2 points)**.** *Using elementary Harmonic Analysis, prove that for every pair of C^1 periodic functions on $[-\pi, \pi]$, one has*

$$\left(\int_{-\pi}^{\pi} h^2 - \dot{h}^2\right)\cdot\left(\int_{-\pi}^{\pi} \psi^2 - \dot{\psi}^2\right) \leq \left(\int h\psi - \dot{h}\dot{\psi}\right)^2.$$

Explain why this provides an alternative solution to Question 2.11 b) on the plane (hint: use Questions 2.23 and 2.16).

Question 2.25 (1 point)**.** *Prove the general version of Brunn's principle: For a convex body $K \subset \mathbb{R}^n$ and a k-dimensional subspace H, the function $|K \cap (y+H)|^{1/k}$ is concave on its support (inside $H^{\perp}$), for $k \in \{1, \ldots, n-2\}$. (The case $k = n-1$ was discussed in class.)*

Question 2.26. *Show that the convolution of log-concave functions is log-concave.*
Hint: Use the fact that marginals of log-concave functions are log-concave, in dimension $\mathbb{R}^{2n}$.

Question 2.27 (1 point)**.** *Provide an alternative proof (to what was done in class) of the Gaussian Poincaré inequality:*

$$\int_{\mathbb{R}^n} f^2 d\gamma - \left(\int_{\mathbb{R}^n} f d\gamma\right)^2 \leq \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma$$

using the decomposition of f into the series of Hermite polynomials (the orthonormal system with respect to the Gaussian measure — you can read about them, e.g., on Wikipedia).

Question 2.28 (1 point)**.** *As per our discussion in class, prove the following statement using the Borell-Brascamp-Lieb inequality (Question 2.9).*

Fix $q \in (-\infty, -n]$. Let $d\mu = e^{-V}dx$ be a probability measure, and let g be a C^1 function. Suppose $V \in C^2(\mathbb{R}^n)$ and $\nabla^2 V - \frac{\nabla V \otimes \nabla V}{q} \geq 0$ (i.e., V is q-concave). Then, assuming all the integrals below exist:

$$\int \left(g e^{\frac{V}{q}}\right)^2 d\mu - \left(\int g e^{\frac{V}{q}} d\mu\right)^2 \leq \frac{-q}{-q+1}\int \left\langle e^{-\frac{2V}{-q}}\left(\nabla^2 V + \frac{\nabla V \otimes \nabla V}{-q}\right)^{-1}\nabla g, \nabla g\right\rangle d\mu.$$

Question 2.29 (1 point). *Deduce the Gaussian Poincaré inequality from the Gaussian Log-Sobolev inequality via the linearization method (this is a partial case of the argument we discussed in class).*

Question 2.30 (1 point). *Prove that the classical Gaussian Log-Sobolev inequality and the classical Lebesgue Log-Sobolev inequality (as stated in class) are equivalent.*

Question 2.31 (1 point). *By differentiating the infimal convolution directly, prove the Gaussian Log-Sobolev inequality without assuming convexity of f:*

$$Ent_\gamma(f^2) \leq 2\int |\nabla f|^2,$$

for any $f \in C^1(\mathbb{R}^n)$ for which the integrals converge.

Question 2.32 (1 point). *Deduce the Sobolev inequality from the Log-Sobolev inequality for the Lebesgue measure.*

Question 2.33 (1 point). *Show that the Gaussian Beckner inequality implies the classical Gaussian Log-Sobolev inequality as $p \to 2$.*

Question 2.34 (1 point). *Deduce Nash's inequality from the classical Lebesgue Log-Sobolev inequality: for any non-negative $f \in L^2(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$,*

$$\left(\int f^2 dx\right)^{1+\frac{2}{n}} \leq \frac{2}{\pi e n}\left(\int |\nabla f|^2 dx\right)\left(\int f dx\right)^{\frac{4}{n}}.$$

Question 2.35 (1 point). *Deduce the isoperimetric inequality from the Sobolev inequality for the Lebesgue measure.*

Question 2.36 (1 point). *Prove the following variant of the Generalized Log-Sobolev inequality: given a log-concave measure μ on $\mathbb{R}^n$ with density e^{-V}, and any pair of smooth convex functions f and g with $\int e^{-f}d\mu = \int e^{-g}d\mu$, one has*

$$\int g^*(\nabla f)e^{-f}d\mu \geq n\int e^{-f}d\mu - \int \langle \nabla V, x\rangle e^{-f}d\mu - \int fe^{-f}d\mu.$$

Question 2.37 (3 points). *Is it possible to obtain Gaussian Beckner inequalities for $p \in [1, 2)$ via linearizations of (some) geometric inequalities directly?*

Question 2.38 (2 points). *Prove the following extension of the Borell-Brascamp-Lieb inequality due to Bolley, Cordero-Erasquin, Fujita, Gentil, and Guillin: for convex f and g on $\mathbb{R}^n$ with $n \geq 2$,*

$$\int (((1-t)f + tg)^*)^{1-n} \geq (1-t)\int (f^*)^{1-n} + t\int (g^*)^{1-n}.$$

Question 2.39 (Generalized Sobolev, 2 points). *Prove the following extension of the Sobolev inequality due to Bolley, Cordero-Erasquin, Fujita, Gentil, and Guillin: for convex F and G on $\mathbb{R}^n$ with $n \geq 2$, such that $\int F^{-n} = \int G^{-n} = 1$, and assuming that $\frac{G(x)}{|x|^\gamma} \to 0$ as $|x| \to \infty$ for some $\gamma > \frac{n}{n-1}$, and that all integrals exist, we have*

$$\int G^*(\nabla F)F^{-n} \geq \frac{1}{n-1}\int G^{1-n}.$$

Question 2.40 (Coredero-Erasquin's proof of Colesanti inequality, 4 points). *Prove the following inequality: when K is a C^2 convex body,* II *is its second fundamental form, and $f \in C^1(\partial K)$ is an arbitrary function such that $\int_{\partial K} f = 0$, then*

$$\int_{\partial K} \mathrm{tr(II)f^2 - \langle II^{-1}\nabla_{\partial K}f, \nabla_{\partial K}f\rangle} \leq 0.$$

Here, $\nabla_{\partial K} f$ stands for the intrinsic boundary gradient of f. Compare to Question 2.11 part b).

Hint: Use the Brascamp-Lieb inequality with $V(x) = \frac{h_K^2(x)}{2}$ and the "body polar coordinates" formula:

$$\int_K F(x)dx = \int_0^\infty \int_{\partial K} F(ty)t^{n-1}\langle y, n_y\rangle dtdy,$$

where n_y is the outer unit normal to ∂K at y, and dy stands for boundary integration.

Question 2.41 (1 point). *Show that when $\varphi : [-\pi, \pi]$ is C^1,* ***even****, and periodic, then*

$$\int_{-\pi}^{\pi} \varphi^2 - \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} \varphi\right)^2 \leq \frac{1}{4}\int_{-\pi}^{\pi} \dot{\varphi}^2.$$

Question 2.42 (1 point). *Show that when $\varphi : [-\pi, \pi]$ is C^1, periodic, and $\varphi(0) = 0$, then*

$$\int_{-\pi}^{\pi} \varphi^2 \leq 4\int_{-\pi}^{\pi} \dot{\varphi}^2.$$

Question 2.43 (1 point). *Show that the Brascamp-Lieb inequality is "the end of the line" for the linearization method: let $d\mu(x) = e^{-V(x)}dx$ and plug the function $f(x) = \langle \nabla V(x), \theta\rangle + \epsilon\varphi$ into Brascamp-Lieb:*

$$\int f^2 d\mu - \left(\int f d\mu\right)^2 \leq \int \langle (\nabla^2 V)^{-1}\nabla f, \nabla f\rangle d\mu,$$

and observe that while $\langle \nabla V(x), \theta\rangle$ indeed attains equality in the above inequality, and the terms corresponding to ϵ cancel out as well, the only inequality that results is again the Brascamp-Lieb inequality.

Question 2.44 (1 point). *Let μ be a log-concave probability measure on $\mathbb{R}^n$ with density e^{-V} for some convex function V, and let the associated Laplacian be $Lu = \Delta u - \langle \nabla u, \nabla V\rangle$. Let $\lambda_1 > 0$ be the first non-trivial eigenvalue of L, i.e., the smallest number such that there exists a non-zero function f_1 with*

$$Lf_1 = -\lambda_1 f_1.$$

Show that

$$\lambda_1 = \inf_{f \in W^{1,2}(d\mu)} \frac{\int |\nabla f|^2 d\mu}{\int f^2 d\mu} = \inf \frac{\int |\nabla f|^2 d\mu}{\int f^2 d\mu - \left(\int f d\mu\right)^2}.$$

Hint: use general convexity/compactness considerations to show the infimum is attained by some function f_1. Then consider $f = f_1 + \epsilon g$ and argue that the derivative with respect to ϵ of that ratio must vanish. Conclude that f has to be an eigenfunction (using general PDE arguments).

Question 2.45 (1 point). *Show that for a positive definite matrix A,*

$$\det(\mathrm{Id} + tA) = 1 + t \cdot \mathrm{tr}(A) + \frac{t^2}{2}\|A\|_{\mathrm{HS}}^2 + o(t^2),$$

where $\|A\|_{\mathrm{HS}}^2$ is the square of the Hilbert-Schmidt norm (i.e., the sum of the squares of all entries).

Question 2.46 (3 points). *Show that one can improve the Gaussian Log-Sobolev inequality to the following: suppose $d\mu = e^{-V - \frac{x^2}{2} - n\log\sqrt{2\pi}}dx = e^{-V}d\gamma$ is a probability measure. Then*

$$-\int V d\mu \le \frac{\int x^2 d\mu - n}{2} + \frac{n}{2}\log\left(2 + \frac{\int |\nabla V|^2 d\mu - \int x^2 d\mu}{n}\right).$$

Question 2.47 (1 point). *Prove the following improvement of the Brascamp-Lieb inequality in the unconditional case (recall that a function $f(x)$ is called unconditional if $f(\epsilon_1 x_1, ..., \epsilon_n x_n) = f(x)$ for every $x \in \mathbb{R}^n$ and choice of signs $\epsilon_i \in \{-1, 1\}$; i.e., f is invariant under coordinate reflections).*

Suppose f, w are unconditional and w is convex. Then for the probability measure $d\mu = Ce^{-w}dx$ one has:

$$\int f^2 d\mu - \left(\int f d\mu\right)^2 \le \int \langle (\nabla^2 w + T)^{-1}\nabla f, \nabla f\rangle d\mu,$$

where $T = \mathrm{diag}\left[\frac{1}{x_1}\frac{\partial w}{\partial x_1}, ..., \frac{1}{x_n}\frac{\partial w}{\partial x_n}\right]$.

Hint: use the multiplicative version of the Prekopa-Leindler inequality for unconditional functions, as in Question 2.10.

Question 2.48 (2 points, important question)**.** *a) Prove the second part of Lemma 7.9 (from the notes) concerning the second derivative of the Legendre transform of an interpolation: for a family of convex functions v_t such that $v_t(x) \in C^2(x,t)$, one has*

$$\frac{d^2}{dt^2} v_t^*(x) = -\ddot{v}_t(\nabla v_t^*) + \left\langle (\nabla^2 v_t(x))^{-1} \nabla \dot{v}_t(\nabla v_t^*), \nabla \dot{v}_t(\nabla v_t^*) \right\rangle .$$

b) Use it to deduce the Brascamp-Lieb inequality from the Prekopa-Leindler inequality directly, without passing through the Generalized Log-Sobolev inequality. Namely, note that the Prekopa-Leindler inequality implies that

$$\frac{d^2}{dt^2} \int e^{-(f+tg)^*} \leq 0,$$

and compute to confirm that this is equivalent to the Brascamp-Lieb inequality

$$\int \varphi^2 d\mu - \left(\int \varphi d\mu \right)^2 \leq \int \langle (\nabla^2 V)^{-1} \nabla \varphi, \nabla \varphi \rangle d\mu,$$

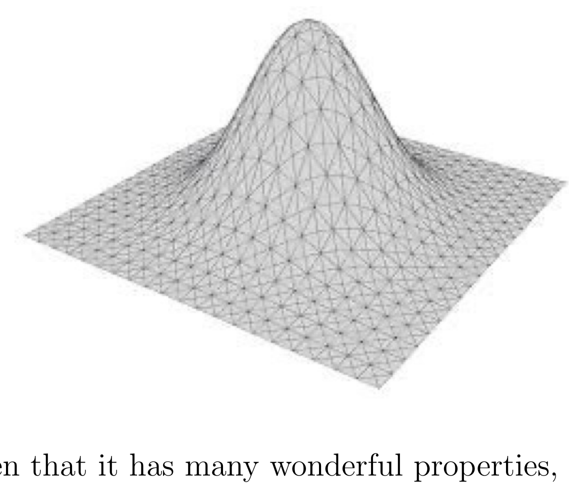
with $d\mu = e^{-V} dx$, where $V = f^*$, and $\varphi(x) = g(\nabla f^*(x))$, assuming $\int d\mu = 1$.

3 Gaussian Measure and its special properties

3.1 A general discussion

Recall that the Gaussian Measure on $\mathbb{R}^n$ is given by

$$d\gamma(x) = \frac{1}{\sqrt{2\pi}^n} e^{\frac{-|x|^2}{2}} dx.$$



We have already seen that it has many wonderful properties, including:

- It is a Log-concave isotropic probability measure; to check the isotropicity, note

$$\int \langle x, \theta \rangle^2 \, d\gamma = \int x_i^2 d\gamma = 1.$$

- It is the only measure both product and rotation invariant
- Linear images of Gaussian random vectors are determined by their Covariance matrix
- Gaussian measure plays the main role in the Central Limit Theorem (and is preserved by convolutions)
- It is extremal for Log-Sobolev inequality
- It is extremal for Reverse Log-Sobolev inequality
- It corresponds to the equality case in the functional Blaschke-Santaló inequality
- It is extremal for the Entropy Power Inequality
- There is a nice "Gaussian Fourier system" called Hermite polynomials
- It satisfies the Poincaré inequality with constant 1
- It satisfies the B-theorem and an improved Poincaré inequality for symmetric functions with constant $\frac{1}{2}$...

It is worthwhile mentioning also:

Theorem 3.1 (Gaussian Correlation Inequality, Royen [29])**.** *If A, B are symmetric convex sets in $\mathbb{R}^n$, then*

$$\gamma(A \cap B) \geq \gamma(A) \cdot \gamma(B).$$

In adition to Royen [29], see also the exposition by Latala, Matlak [24] in regards to the above breakthrough result.

The following Proposition is a way to quantify that *the Gaussian measure is "a role model" for all isotropic probability measures (and especially for log-concave ones).* Recall $C_p(\mu)$, the Poincare constant associated with a measure μ, is the smallest number such that for all f:

$$\int f^2 d\mu - \left(\int f d\mu \right)^2 \leq C_p(\mu) \int |\nabla f|^2 \, d\mu.$$

Proposition 3.1. *Suppose μ is an isotropic probability measure. Then*

$$C_p(\mu) \geq 1 = C_p(\gamma).$$

Proof. Recall the fact that μ is isotropic implies that $\int x d\mu = 0$, i.e. $\forall i \int x_i d\mu = 0$. Also $\forall \theta \in S^{n-1}, \int \langle x, \theta \rangle^2 d\mu = 0$, and in particular

$$\int x_i^2 d\mu = 1.$$

So,

$$1 = \int x_i^2 d\mu - \left(\int x_i d\mu \right)^2 \leq C_p(\mu) \int 1 d\mu = C_p(\mu)$$

□

In this section, we shall see several very strong isoperimetric-properties and phenomena which are unique to the Gaussian measure.

3.2 The isoperimetric profile

Recall the Isoperimetric problem for general (probability) measures μ. The objective of this problem is to find

$$\inf_{\mu(A)=a} \mu^+(\partial A)$$

for a given $a \in [0, 1]$, where the weighted perimeter is defined as

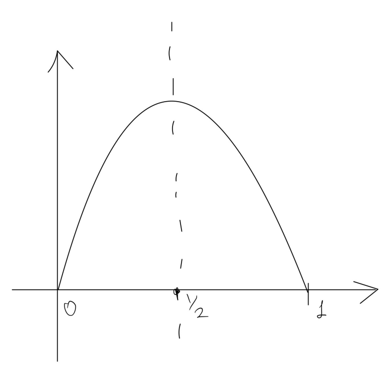
$$\mu^+(\partial A) = \liminf_{\varepsilon \to 0} \frac{\mu(A + \varepsilon B_2^n \backslash A)}{\varepsilon}.$$

Definition 3.1. The isoperimetric profile of μ is defined as

$$I_\mu(a) = \inf_{\mu(A)=a} \mu^+(\partial A).$$

Below are some properties of $I_\mu(a)$.

- For non-atomic measures, $I_\mu(a) \geq 0$, $I_\mu(a) \to 0$ as $a \to 0$, and $I_\mu(a) \to 0$ as $a \to 1$
- $I_\mu(a - \frac{1}{2})$ is even. One can see this by taking complements, i.e. $\mu(A) = 1 - \mu(A^c)$ but they have the same perimeters $\mu^+(\partial A) = \mu^+(\partial A^c)$.
- I_μ is convex for log-concave measures (proved by E. Milman [26])



Remark 3.1. *Consider for example the Lebesgue measure $|A| = a$, then $|\partial A| \geq c_n \cdot a^{1/n}$, and we have concavity. Recall that the proof of this followed from the Brunn-Minkowski inequality.*

So to obtain an isoperimetry bound for a general log-concave measure, one may try to use the Prekopa-Leindler inequality.

$$\begin{aligned}
\mu^+(\partial K) &= \liminf_{\varepsilon\to 0} \frac{\mu(K+\varepsilon B_2^n) - \mu(K)}{\varepsilon} \\
&= \liminf_{\varepsilon\to 0} \frac{\mu\left((1-t)\frac{K}{1-t} + t\frac{\varepsilon B_2^n}{t}\right) - \mu(K)}{\varepsilon} \\
&\geq \sup_t \liminf_{\varepsilon\to 0} \frac{\mu\left(\frac{K}{1-t}\right)^{1-t}\mu\left(\frac{\varepsilon B_2^n}{t}\right)^t - \mu(K)}{\varepsilon}.
\end{aligned}$$

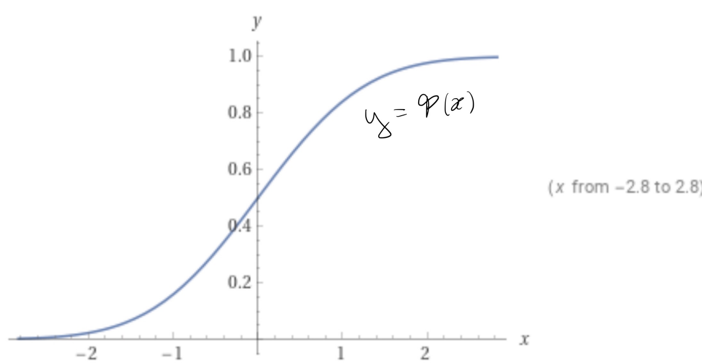
However, one cannot hope to get a sharp bound because the Prekopa-Leindler inequality is never tight!

3.3 The Ehrard Inequality

The Ehrard inequality is a fancier and tighter Gaussian version of the Prekopa-Leindler inequality.

Definition 3.2. The Gaussian cumulative distribution function (c.d.f.) is denoted as

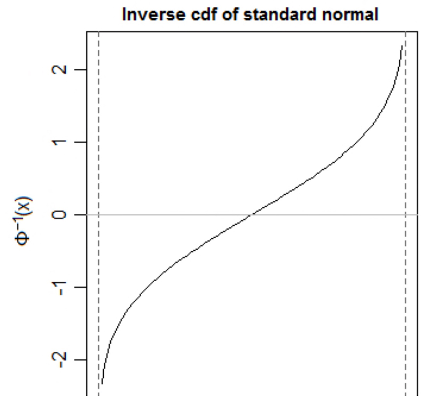
$$\Phi(s) = \int_{-\infty}^{s} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = \gamma_1(-\infty, s)$$



Note that

$$\Phi'(s) = \frac{1}{\sqrt{2\pi}} e^{-s^2/2}.$$

We will be considering the inverse $\Phi^{-1} : [0,1] \to \mathbb{R}$.



We are ready to state arguably the most important Gaussian Isoperiemtric-type inequality, proved by Ehrhard for convex sets [15], [14] then by Latala when one of the sets is convex [22], and finally by Borell [10], [11] for arbitrary Borel-measurable sets:

Theorem 3.2 (Ehrard-Borell)**.** *For all Borel measurable sets $A, B \subset \mathbb{R}^n$, $\forall\ \lambda \in (0,1)$,*

$$\Phi^{-1}(\gamma(\lambda A + (1-\lambda)B)) \geq \lambda\Phi^{-1}(\gamma(A)) + (1-\lambda)\Phi^{-1}(\gamma(B)).$$

Other proofs are given by Neeman and Paouris [28], van Handel [18] and Ivanisvili [20], among others.

Let us compare this to Prekopa-Leindler. Gaussian measure is log-concave, which means that

$$\log\gamma(\lambda A + (1-\lambda)B) \geq \lambda\log(\gamma(A)) + (1-\lambda)\log(\gamma(B))$$

Consider the function

$$m(t) = \gamma((1-t)A + tB).$$

Ehrard's inequality says that $\Phi^{-1}\circ m$ is concave, while Prekopa-Leindler says that $\log m$ is concave.

Claim 3.1. *Ehrhard's inequality is stronger than the Prekopa-Leindler inequality for the Gaussian measure: for any strictly increasing function m $\Phi^{-1}\circ m$ is concave implies that $\log m$ is concave*

Proof. We consider the local form of the functions. Notice that $\Phi^{-1}\circ m$ is equivalent to the fact that

$$(\Phi^{-1}\circ m)'' \leq 0,$$

and $\log\circ m$ is concave is equivalent to the fact that

$$(\log\circ m)'' \leq 0.$$

In general, we have

$$(f\circ m)'' = f''\cdot(m')^2 + f'm'' \leq 0,$$

which is true if and only if

$$\frac{m''}{(m')^2} \leq -\frac{f''}{f'}$$

for $f' > 0$. So relating this back to Ehrard, we have

$$\frac{m''}{(m')^2} \leq -\frac{(\Phi^{-1})''}{(\Phi^{-1})'},$$

and for Prekopa-Leindler, we have

$$\frac{m''}{(m')^2} \leq -\frac{(\log)''}{(\log)'}.$$

To prove that Ehrard is stronger than Prekopa-Leindler, we prove the following key claim

$$\forall t : -\frac{(\Phi^{-1}(t))''}{(\Phi^{-1}(t))'} \leq -\frac{(\log(t))''}{(\log(t))'}. \tag{13}$$

We compute each component of this inequality

$$\frac{(\log(t))''}{(\log(t))'} = -\frac{1/t^2}{t/t} = -\frac{1}{t}$$

$$(\Phi^{-1}(t))' = \frac{1}{\Phi'(\Phi^{-1}(t))} = \sqrt{2\pi}e^{\Phi^{-1}(t)^2/2}$$

$$(\Phi^{-1}(t))'' = 2\pi\Phi^{-1}(t) \cdot e^{\Phi^{-1}(t)^2/2}$$

$$\implies \frac{(\Phi^{-1}(t))''}{(\Phi^{-1}(t))'} = \sqrt{2\pi}\Phi^{-1}(t)e^{\Phi^{-1}(t)^2/2}.$$

The fact that 13 implies the overall claim is equivalent to saying that

$$-\sqrt{2\pi}\Phi^{-1}(t)e^{\Phi^{-1}(t)^2/2} \geq \frac{1}{t}. \tag{14}$$

Why is 14 true? Indeed, if $a = \Phi^{-1}(t)$, then (14) becomes

$$-\frac{1}{\Phi(a)} \leq \sqrt{2\pi}ae^{a^2/2}$$

If $a \geq 0$, the inequality is trivially true. If $a \leq 0$, we have to show that

$$\int_{-\infty}^{a} e^{-t^2/2}dt \leq -\frac{1}{a}e^{-a^2/2}.$$

Change of variables $b = -a$ gives us

$$\int_{b}^{\infty} e^{-t^2/2}dt \leq \frac{1}{b}e^{-b^2/2}$$

when $b \geq 0$. To see why the above inequality is true, notice that

$$\begin{aligned}
\int_b^\infty e^{-t^2/2} dt &= \int_b^\infty t \cdot \frac{1}{t} e^{-t^2/2} dt \\
&\leq \frac{1}{b} \int_b^\infty t e^{-t^2/2} dt \\
&= \frac{1}{b} e^{-b^2/2}
\end{aligned}$$

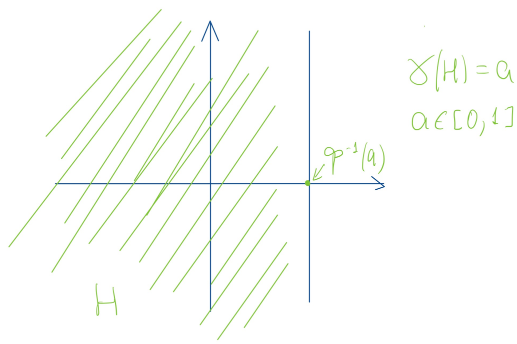
So the claim is proved. $\square$

Remark 3.2. *What is the geometric meaning of $\Phi^{-1}(a)$, for $a \in [0,1]$. Consider the half space*

$$H = \{x \in \mathbb{R}^n : x_1 \leq \alpha\}.$$

Then $\gamma(H) = a$ means that $\Phi^{-1}(a) = \alpha$. Indeed,

$$\Phi(\alpha) = \gamma_1(-\infty, \alpha) = \gamma_n(H) = a.$$



Compare this to $f(t) = t^{1/n}$ - the (multiple of) the radius of the ball of Lebesgue volume t (see the below remark).

Remark 3.3. *van Handel, Shenfeld [19] fully characterized the equality cases in Ehrhard's inequality.*

In particular, the equality in the Ehrhard inequality is attained when A, B are parallel half-spaces. Indeed,

$$\begin{aligned}
A &= \{x \in \mathbb{R}^n : x_1 \leq \Phi^{-1}(a)\}, \ \gamma(A) = a \\
B &= \{x \in \mathbb{R}^n : x_1 \leq \Phi^{-1}(b)\}, \ \gamma(B) = b
\end{aligned}$$

Then

$$\frac{A+B}{2} = \left\{ x \in \mathbb{R}^n : x_1 \leq \frac{\Phi^{-1}(a) + \Phi^{-1}(b)}{2} \right\}.$$

So

$$\Phi^{-1}\left(\gamma\left(\frac{A+B}{2}\right)\right) = \frac{\Phi^{-1}(\gamma(A)) + \Phi^{-1}(\gamma(B))}{2} = \frac{\Phi^{-1}(a) + \Phi^{-1}(b)}{2}$$

the point where $\frac{A+B}{2}$ intersects the x-axis.

3.4 Gaussian isoperimetric inequality

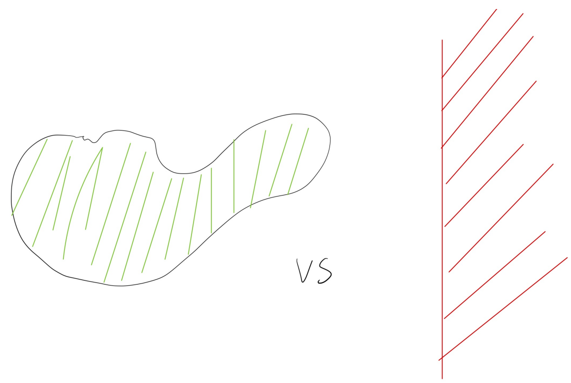
Ehrhard's inequality implies the following classical and important result:

Theorem 3.3 (The Gaussian Isoperimetric inequality, Sudokov-Tsirelson [32], Borell [11]). *If A is a Borel measurable in $\mathbb{R}^n$ with $\gamma(A) = a \in [0,1]$, then*

$$\gamma^+(\partial A) \geq \gamma^+(\partial H_a) = \frac{1}{\sqrt{2\pi}} e^{-\Phi^{-1}(a)^2/2}$$

where H_a is the halfspace of measure a

$$H_a = \{x \in \mathbb{R}^n : x_1 \leq \Phi^{-1}(a)\}.$$



In other words,

$$I_\gamma(a) = \frac{1}{\sqrt{2\pi}} e^{-\Phi^{-1}(a)^2/2}.$$

Note that this implies that

$$I_\gamma(a) = \frac{1}{\Phi^{-1}(a)'},$$

and that

$$I_\gamma(a) \cdot I''_\gamma(a) = -1.$$

Proof. (of the Gaussian isoperimetry via Ehrhard) Let K be some Borel set in $\mathbb{R}^n$. Then

$$\begin{aligned}
\gamma^+(\partial K) &= \liminf_{\varepsilon\to 0} \frac{\gamma(K+\varepsilon B_2^n) - \gamma(K)}{\varepsilon} \\
&= \sup_{\lambda>0} \liminf_{\varepsilon\to 0} \frac{\gamma\left((1-\lambda)\frac{K}{1-\lambda} + \lambda \frac{\varepsilon B_2^n}{\lambda}\right) - \gamma(K)}{\varepsilon} \\
&\geq \liminf_{\varepsilon\to 0,\ \lambda\to 0} \frac{\Phi\left((1-\lambda)\Phi^{-1}\left(\frac{K}{1-\lambda}\right) + \lambda\Phi^{-1}\left(\frac{\varepsilon B_2^n}{\lambda}\right)\right) - \gamma(K)}{\varepsilon},
\end{aligned}$$

where the last inequality follows from Ehrard's inequality. Now let $t = \frac{\varepsilon}{\lambda}$. t can be anything since ε and λ can tend to 0 at different rates. It turns out the optimal case is taking $t \to \infty$. Since $\gamma(K) = a$, the last line of the above becomes (by Taylor's Theorem)

$$
\begin{aligned}
&= \lim_{t\to\infty} \Phi'(\Phi^{-1}(a)) \cdot \frac{\Phi^{-1}(\partial(tB_2^n))}{t} \\
&= \lim_{t\to\infty} \Phi'(\Phi^{-1}(a)) \\
&= \frac{1}{\sqrt{2\pi}} e^{-\Phi^{-1}(a)^2/2} \\
&= \gamma^+(\partial H_a).
\end{aligned}
$$

The fact that $\lim_{t\to\infty} \frac{\Phi^{-1}(\partial(tB_2^n))}{t} = 1$ is left as homework. $\square$

One can then ask the question about an anisotropic version of this. In other words, instead of taking B_2^n, take some set L (this is left as homework).

3.5 Gaussian concentration inequality and Borell's noise stability

Theorem 3.4 (Gaussian concentration inequality and Borell's noise stability)**.**

$$\gamma(A_t) \geq 1 - \frac{1}{2} e^{-t^2/2}$$

if $\gamma(A) \geq 1/2$*. Moreover, if* H *is a half-space with* $\gamma(A) = \gamma(H) = a \in [0,1]$*, we have*

$$\gamma(A_t) \geq \gamma(H_t) = \Phi(\Phi^{-1}(a) + t).$$

Proof. Let

$$h(t) = \Phi^{-1}(\gamma(A_t)).$$

Note that

$$h'(t) = \sqrt{2\pi} e^{\frac{\Phi^{-1}(\gamma(A_t))^2}{2}} \cdot \frac{d}{dt}\gamma(A_t) \geq \frac{\gamma^+(\partial A_t)}{I_\gamma(\gamma(A_t))} \geq 1.$$

Above, the second to last inequality follows from the Gaussian isoperimetric inequality, and the last inequality follows from the definition of the isoperimetric profile. We will now apply Newton's formula

$$h(t) = h(0) + \int_0^t h'(s)ds \geq h(0) + \int_0^t ds = h(0) + t$$

$$\implies \Phi'(\gamma(A_t)) \geq \Phi^{-1}(\gamma(A)) + t,$$

which implies

$$\gamma(A_t) \geq \Phi(\Phi^{-1}(\gamma(A)) + t) = \gamma(H_t),$$

where $\gamma(A) = \gamma(H)$.

Next, suppose $\gamma(A) \geq 1/2$. Then $\Phi^{-1}(\gamma(A)) \geq 0$, and

$$\gamma(A_t) \geq \Phi(\Phi^{-1}(\gamma(A)) + t) \geq \Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{\infty} e^{-s^2/2} ds,$$

or

$$1 - \gamma(A_t) \leq \frac{1}{\sqrt{2\pi}} \int_{t}^{\infty} e^{-s^2/2} ds.$$

It now suffices to show

$$\frac{1}{\sqrt{2\pi}} \int_{t}^{\infty} e^{-s^2/2} ds \leq \frac{1}{2} e^{-t^2/2}.$$

Consider for $t > 0$,

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_{t}^{\infty} e^{-s^2/2} ds - \frac{1}{2} e^{-t^2/2},$$

and thus

$$F'(t) = -\frac{1}{\sqrt{2\pi}} e^{-t^2/2} - \frac{t}{2} e^{-t^2/2}$$

and

$$F''(t) = \frac{t}{\sqrt{2\pi}} e^{-t^2/2} - \frac{1}{2} e^{-t^2/2} + \frac{t^2}{2} e^{-t^2/2}.$$

Observe that

$$F'\left(\sqrt{\frac{2}{\pi}}\right) = 0,$$

and

$$F''(t) \geq 0 \text{ if and only if } t \geq \sqrt{\frac{2}{\pi}}.$$

Note also that

$$F \to 0 \text{ as } t \to \infty,$$

and that $F(0) = 0$. This means that F is concave and non-decreasing on $[0, \sqrt{\frac{2}{\pi}}]$ (and thus is non-negative on that interval), and convex and non-increasing on $[\sqrt{\frac{2}{\pi}}, \infty]$, and as it also tends to zero at infinity, we conclude that it must remain non-negative. This concludes the proof. $\square$

3.6 Gaussian symmetrization and the proof of the Ehrhard inequality

The following concept was introduced by Ehrhard [15], [14], see also Borell [10] and Bogachev [7].

Definition 3.3 (Gaussian symmetrization)**.** Fix an integer k, $1 \le k \le n$. Fix also L, a subspace of $\mathbb{R}^n$ with $\dim L = n-k$. Fix any $e \perp L$. Then for a Borel measurable set $A \subset \mathbb{R}^n$, consider the Gaussian symmetrization of A denoted by

$$S(L,e)(A)$$

such that for all $x \in L$

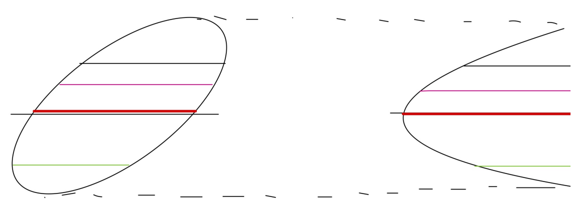
$$S(L,e)(A) \cap (x + L^{\perp}) = \{y : \langle y, e\rangle \ge r\} \cap (x + L^{\perp})$$

where $r = r(x)$ is chosen so that

$$\gamma_k(S(L,e)(A) \cap (x + L^{\perp})) = \gamma_k(A \cap (x + L^{\perp})).$$

Here are some examples:

- If $L = e_1^{\perp}$ and $e = e_1$, then $S(e_1^{\perp}, e_1)(A) = \left\{x \in \mathbb{R}^n : x_1 \le \Phi_1^{-1}(\gamma(A))\right\}$;
- If $k = n-1$, then this corresponds to matching $(n-1)$-dimensional slices of set A to rays J of a 2-dimensional set.



Some properties of the Gaussian symmetrization are left as homework:

Lemma 3.1. *Let A and B be Borel-measurable sets in $\mathbb{R}^n$ and $v \in \mathbb{R}^n$. Then*

- $\gamma(S(L,e)(A)) = \gamma(A)$;
- $A \subset B \implies S(L,e)(A) \subset S(L,e)(B)$;
- $S(L,e)(A+v) = S(L,e)(A) + v$.

Also,

Lemma 3.2. *Let L_1, L_2 be linear subspaces such that $(L_1 \cap L_2)^{\perp} \cap L_1$ and $(L_1 \cap L_2)^{\perp} \cap L_2$ are orthogonal. Then*

$$S(L_1,e) \circ S(L_2,e) = S(L_2,e) \circ S(L_1,e) = S(L_1 \cap L_2, e)$$

This implies:

Corollary 3.1. *Let $n \geq 3$ and $k \geq 2$. Then for all k-symmetrizations $S = S(L, e)$, there is a sequence of 2-symmetrizations $S_1, \ldots, S_{k-1}$ such that $S = S_1 \circ \ldots \circ S_{k-1}$.*

Furthermore,

Lemma 3.3. *In dimension 2, there exists a sequence $\theta_1, \ldots, \theta_k, \ldots$ such that*

$$S(\theta_k^\perp, \theta_k) \circ S(\theta_{k-1}^\perp, \theta_{k-1}) \circ \ldots \circ S(\theta_1^\perp, \theta_1)(A)$$

converges in Hausdorff distance to a half-space of the same Gaussian measure as A.

Combining Corollary 3.1 and Lemma 3.3 we get

Corollary 3.2. *The statement of Lemma 3.3 is true in any dimension.*

Remark 3.4. *If A is a half-space, then it is invariant under any symmetrizations.*

Next, we formulate:

Theorem 3.5. *If A is a closed set in $\mathbb{R}^n$, then for all L and for all $e \in L^\perp$,*

$$S(L, e)(A) + rB_2^n \subset S(L, e)(A + rB_2^n).$$

Proof. Home work! □

Remark 3.5. *The previous Theorem implies the Gaussian isoperimetric inequality, when combined with Corolalry 3.2 (without going via Ehrhard's inequality).*

Finally, the following result is crucial in our proof of Ehrhard's inequality, and its proof is based on all the results above, and is left as a home work:

Theorem 3.6. *If A is an open convex set in $\mathbb{R}^n$, then for all L and for all $e \in L^\perp$, then $S(L, e)(A)$ is convex.*

Proof. Home work! □

We are now ready to prove the Ehrhard inequality **for convex sets** A **and** B. Recall that it states that for any $\lambda \in [0, 1]$,

$$\Phi^{-1}\left(\gamma\left(\lambda A + (1-\lambda)B\right)\right) \geq \lambda\Phi^{-1}(\gamma(A)) + (1-\lambda)\Phi^{-1}(\gamma(B)). \tag{15}$$

The idea is to consider n-dimensional convex sets A and B as parallel sections of an $(n+1)$-dimensional convex set, symmetrize it into a 2-dimensional convex set, and the convexity of this set (which follows from Theorem 3.6) is exactly the statement of Ehrhard's inequality (15).

Proof. (of Ehrhard's inequality for convex sets.)

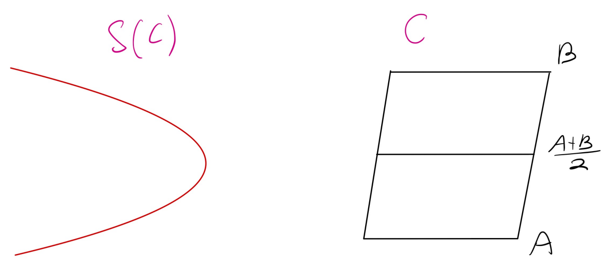
Consider A, B as subsets of $\mathbb{R}^{n+1}$: let

$$\widetilde{A} = A \times \{0\},$$

$$\widetilde{B} = B \times \{0\}$$

and $C = \operatorname{conv}(\widetilde{A}, \widetilde{B})$. Then

$$C \cap \left\{ e_{n+1}^{\perp} + \frac{1}{2} e_{n+1} \right\} = \frac{A+B}{2} \cap \left\{ e_{n+1}^{\perp} + \frac{1}{2} e_{n+1} \right\}$$



Take n-symmetrizations in $\mathbb{R}^{n+1}$ of C such that intersections with n-dimensional hyperplanes are preserved. Let

$$C_\lambda = e_{n+1}^{\perp} \cap (C - \lambda e_{n+1}) = e_{n+1}^{\perp} \cap (\lambda A + (1-\lambda) B)$$

and

$$f(\lambda) = \Phi^{-1}(\gamma(C_\lambda))$$

Then by definition of symmetrization

$$(\lambda e_{n+1} + e_{n+1}^{\perp}) \cap S(C) = (e_{n+1} + e_{n+1}^{\perp}) \cap \{x \in \mathbb{R}^n : \langle x, e \rangle \geq r\}$$

where $r = -f(\lambda)$. By Theorem 3.6, the set $S(C)$ is convex, or equivalently $f(\lambda)$ is concave. Therefore,

$$\Phi^{-1}(\gamma(C_\lambda)) \text{ is convex,}$$

yielding

$$\iff \Phi^{-1}(\gamma(\lambda A + (1-\lambda)B)) \geq \lambda \Phi^{-1}(\gamma(A)) + (1-\lambda)\Phi^{-1}(\gamma(B)).$$

$\square$

3.7 The Latała's Functional Ehrhard inequality

In this subsection, we present the functional version of the Ehrhard inequality which was observed by Latała [23]

Theorem 3.7 (Functional Ehrhard's inequality, Latała [23]). *Let $\lambda \in [0,1]$, and suppose $F, G, H : \mathbb{R}^n \to [0,1]$ are such that for all $x, y \in \mathbb{R}^n$,*

$$\Phi^{-1}(H(\lambda x + (1-\lambda)y)) \geq \lambda \Phi^{-1}(F(x)) + (1-\lambda)\Phi^{-1}(G(y)). \tag{16}$$

Then

$$\Phi^{-1}\left(\int_{\mathbb{R}^n} H d\gamma\right) \geq \lambda \Phi^{-1}\left(\int_{\mathbb{R}^n} F d\gamma\right) + (1-\lambda)\Phi^{-1}\left(\int_{\mathbb{R}^n} G d\gamma\right).$$

Therefore, for convex f, g,

$$\Phi^{-1}\left(\int \Phi(-(\lambda f + (1-\lambda)g)^* d\gamma\right) \geq \lambda \Phi^{-1}\left(\int \Phi(-f^*) d\gamma\right) + (1-\lambda)\Phi^{-1}\left(\int \Phi(-g^*) d\gamma\right).$$

In other words, $\Phi^{-1}\left(\int \Phi(-(f+tg)^* d\gamma\right)$ is concave. Here, as before, f^* stands for Legendre transform.

Remark 3.6. *As before, one may note that $(\lambda f^* + (1-\lambda)g^*)^* = f \Box_\lambda g$ satisfies (16) and this is why one can reformulate it in terms of Legendre transform.*

Proof. Consider $A, B \subset \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ given by

$$A = \left\{(x,y) : y \leq \Phi^{-1}(F(x))\right\}$$
$$B = \left\{(x,y) : y \leq \Phi^{-1}(G(x))\right\}.$$

A and B are subgraphs, and $x \in \mathbb{R}^n$, $y \in \mathbb{R}$. Then the condition of the theorem implies

$$\lambda A + (1-\lambda)B \subset \left\{(x,y) : \ y \leq \Phi^{-1}(H(x))\right\} \subset \mathbb{R}^{n+1}. \tag{17}$$

Ehrhard's inequality in $\mathbb{R}^{n+1}$ implies

$$\Phi^{-1}\left(\gamma(\lambda A + (1-\lambda)B)\right) \geq \lambda \Phi^{-1}(\gamma(A)) + (1-\lambda)\Phi^{-1}(\gamma(B)). \tag{18}$$

Then (17) and (18) imply

$$\Phi^{-1}(\gamma((x,y) : y \leq \Phi^{-1}(H))) \geq$$
$$\lambda \Phi^{-1}(\gamma((x,y) : y \leq \Phi^{-1}(F))) + (1-\lambda)\Phi^{-1}(\gamma((x,y) : y \leq \Phi^{-1}(G))).$$

This implies the desired inequality, in view of the fact that by Fubini

$$\gamma((x,y) : y \leq \Phi^{-1}(F)) = \int F d\gamma.$$

$\square$

Remark 3.7. *Functional Ehrhard also tensorizes (this is left as homework). But the base case of the induction (the 1-dimensional case) is difficult.*

3.8 Generalized Bobkov's inequality via linearizing functional Ehrhard's inequality

In this subsection we will do the same procedure with Ehrhard's inequality that allowed us to deduce the Generalized Log-Sobolev inequality from the Prekopa-Leindler inequality, following Barthe, Cordero-Erausquin, Ivanisvili, Livshyts [5]. We remark that an alternative procedure which involved linearization of the geometric Ehrhard inequality directly (rather than its functional version) was done by Kolesnikov and Milman [21], and a number of interesting geometric corollaries was obtained. It remains unclear if there are direct links between the work in [21] and what we are about to present.

Consider

$$\begin{aligned}\alpha(t) = \Phi^{-1}\left(\int \Phi(-((1-t)f+tg)^* d\gamma\right) - (1-t)\Phi^{-1}\left(\int \Phi(-f^*)d\gamma\right)\\ - t\Phi^{-1}\left(\int \Phi(-g^*)d\gamma\right).\end{aligned}$$

Then Functional Ehrhard's inequality Theorem 3.7 implies

$$\alpha(t) \geq 0 \text{ for all } t \in [0,1]$$

and

$$\begin{aligned}\alpha(0) = 0\\ \implies \alpha'(0) \geq 0.\end{aligned}$$

Recall

$$\Phi'(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}},$$

and

$$\frac{d}{da}\Phi^{-1}(a) = \frac{1}{\Phi'(\Phi^{-1}(a))} = \sqrt{2\pi} e^{\frac{\Phi^{-1}(a)^2}{2}} = \frac{1}{I(a)},$$

where

$$I(a) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\Phi^{-1}(a)^2}{2}}$$

is the Gaussian isoperimetric profile. Recall that $\frac{d}{dt}((1-t)f+tg)^* = (f-g)(\nabla f^*)$ (see Lemma 2.2). We then write

$$\begin{aligned}\alpha'(0) = \frac{1}{I\left(\int \Phi(-f^*)\right)} \cdot \int \frac{1}{\sqrt{2\pi}} e^{-\frac{f^{*2}}{2}} \cdot -(f(\nabla f^*) - g(\nabla f^*))d\gamma\\ + \Phi^{-1}\left(\int \Phi(-f^*)d\gamma\right) - \Phi^{-1}\left(\int \Phi(-g^*)d\gamma\right)\\ \geq 0.\end{aligned}$$

So we have

$$\spadesuit(f) + \int g(\nabla f^*) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{f^{*2}}{2}} d\gamma \geq I \quad \left(\int \Phi(-f^*) d\gamma \right) \cdot \Phi^{-1} \left(\int \Phi(-g^*) d\gamma \right),$$

where $\spadesuit(f)$ is a function that depends on only f and not g. Set $G = g^*$, $f^* = -\Phi^{-1}(h)$ for function h. Then

$$\nabla f^* = -\frac{\nabla h}{I(h)},$$

$$I(h) = \frac{1}{\sqrt{2\pi}} e^{-\frac{f^{*2}}{2}},$$

$$\spadesuit(h) + \int G^* \left(-\frac{\nabla h}{I(h)} \right) \cdot I(h) d\gamma \geq I \quad \left(\int h d\gamma \right) \cdot \Phi^{-1} \left(\int \Phi(-G) d\gamma \right).$$

Recall

$$\begin{aligned} \left(\lambda G \left(\frac{x}{\lambda} \right) \right)^* \Big|_z &= \sup_y \left(\langle y, z \rangle - \lambda G \left(\frac{y}{\lambda} \right) \right) \\ &= \lambda \sup_t \left(\langle t, z \rangle - G(t) \right) \\ &= \lambda G^*(z). \end{aligned}$$

where we did a change of variables $t = \frac{y}{\lambda}$. Then for every λ,

$$\spadesuit(h) + \int G^* \left(-\frac{\nabla h}{I(h)} \right) \cdot I(h) d\gamma \geq I \quad \left(\int h d\gamma \right) \cdot \Phi^{-1} \left(\int \Phi \left(-\lambda G \left(\frac{x}{\lambda} \right) \right) d\gamma \right).$$

We divide both sides by λ and let $\lambda \to \infty$. Note that $\frac{\spadesuit(h)}{\lambda} \to 0$, and we get:

Theorem 3.8 ("Generalized Bobkov's inequality", Barthe, Cordero-Erasquin, Ivanisvili, Livshyts [5])**.** *For all convex G and for all h (such that the integrals make sense)*

$$\int G^* \left(-\frac{\nabla h}{I(h)} \right) \cdot I(h) d\gamma \geq I \left(\int h d\gamma \right) \cdot \lim_{\lambda \to \infty} \frac{\Phi^{-1} \left(\int \Phi \left(-\lambda G \left(\frac{x}{\lambda} \right) \right) d\gamma \right)}{\lambda}.$$

Remark 3.8. *If G is ray-increasing, we have*

$$G \left(\frac{x}{\lambda} \right) \geq G(0).$$

Thus

$$\begin{aligned} \lim_{\lambda \to \infty} \frac{\Phi^{-1} \left(\int \Phi \left(-\lambda G \left(\frac{x}{\lambda} \right) \right) d\gamma \right)}{\lambda} &\leq \lim_{\lambda \to \infty} \frac{\Phi^{-1} \left(\int \Phi \left(-\lambda G(0) \right) d\gamma \right)}{\lambda} \\ &= \lim_{\lambda \to \infty} \frac{-\lambda G(0)}{\lambda} \\ &= -G(0). \end{aligned}$$

In fact, often $\geq$ holds as well. Note that ray increasing means that $\forall t > 0$, $\forall \theta \in S^{n-1}$, $G(t\theta)$ is increasing in t.

We will consider the following example

$$G(x) = \begin{cases} -\sqrt{1-|x|^2} & \text{if } |x| \le 1 \\ \infty & \text{if } |x| > 1 \end{cases}.$$

$$\Phi\left(-\lambda G\left(\frac{x}{\lambda}\right)\right) d\gamma = \int_{\lambda B_2^n} \Phi\left(\sqrt{\lambda^2 - x^2}\right) d\gamma \approx \Phi(\lambda),$$

and

$$\lim_{\lambda\to\infty} \frac{\Phi^{-1}\left(\Phi(\lambda)\right)}{\lambda} = 1.$$

We leave the details of this limit as a home work.

Recall (Example 2.2, part 5) that $G^*(x) = \sqrt{1+|x|^2}$. Note that

$$I(h)G^*\left(\frac{\nabla h}{I(h)}\right) \ge I(h)\sqrt{1 + \frac{|\nabla h|^2}{I(h)^2}} = \sqrt{I(h)^2 + |\nabla h|^2}.$$

Plugging this G into Theorem 3.8 we deduce the following celebrated inequality of Bobkov (which was originally proved via different means).

Theorem 3.9 (Bobkov [6])**.**

$$\int_{\mathbb{R}^n} \sqrt{I(h)^2 + |\nabla h|^2} \ge I\left(\int_{\mathbb{R}^n} h d\gamma\right).$$

The inequality tenzorizes, so one can use induction in dimension and so-called 2-point symmetrizations for the proof, as was done in [6]. Several alternative proofs were given by Barthe, Ivanisvili [4], Carlen, Kerce [12], Neeman, Paouris [28], among others. The proof that was presented in these notes is by Barthe, Cordero-Erausquin, Ivanisvili, Livshyts [5].

Remark 3.9. *Bobkov's inequality is implies (and in fact follows from) the Gaussian isoperimetric inequality. Indeed, let $h = \mathbf{1}_K$ and so $|\nabla h| = h_x \mathbf{1}_{\{x\in\partial K\}}$. The LHS of the inequality is $\gamma^+(\partial K)$ and the RHS is $I(\gamma(K))$. So we have*

$$\gamma^+(\partial K) \ge I(\gamma(K)) = \gamma^+(\partial H),$$

where H is a half-space and $\gamma(H) = \gamma(K)$. See e.g. Neeman [27] for the opposite implication.

So we get the following "diagram":

$$\begin{array}{rcr}
\text{Ehrhard} & \longrightarrow & \text{Prekopa-Leindler} \\
\downarrow & & \downarrow \\
\text{Generalized Bobkov} & \longrightarrow & \text{Generalized log-Sobolev} \\
\downarrow & & \downarrow \\
\text{Bobkov} & \longrightarrow & \text{Log-Sobolev} \\
\updownarrow & & \updownarrow \\
\text{Gaussian Isoperimetry} & & \text{Classical isoperimetry}
\end{array}$$

Consider now another example:

$$G(x) = \begin{cases} -1 & \text{if } \|x\|_K \le 1 \\ \infty & \text{if } \|x\|_K > 1 \end{cases}$$

Then $G^*(x) = 1 + h_K(x) = 1 + \|x\|_{K^o}$, and we get:

Corollary 3.3.

$$I\left(\int h d\gamma\right) \cdot \lim_{\lambda\to\infty} \frac{\Phi^{-1}(\Phi(\lambda)\gamma(\lambda K))}{\lambda} \le \int |\nabla h| d\gamma + \int I(h) d\gamma.$$

In particular, for $K = B_2^n$

$$I\left(\int h d\gamma\right) - \int I(h) d\gamma \le \int |\nabla h| d\gamma.$$

Remark 3.10. *The last inequality is weaker than Bobkov's inequality since*

$$\int_{\mathbb{R}^n} \sqrt{I(h)^2 + |\nabla h|^2} \le I\left(\int h d\gamma\right) - \int I(h) d\gamma.$$

Remark 3.11. *More generally for*

$$G(x) = \begin{cases} -\sqrt[p]{1-|x|^p} & \textit{if } \|x\|_K \le 1 \\ \infty & \textit{if } \|x\|_K > 1 \end{cases},$$

one can obtain p-Bobkov inequalities.

3.9 An Ehrhard-Brascamp-Lieb type inequality

We will now differentiate Ehrhard's inequality twice to obtain a version of Ehrhard-Brascamp-Lieb inequality. Note that Theorem 3.7 implies that

$$\frac{d^2}{dt^2}\Phi^{-1}\left(\int \Phi(-(f+tg)^*) d\gamma\right) \le 0 \tag{19}$$

The left hand side of the above equals to

$$\frac{d}{dt}\left[\frac{1}{I\left(\int \Phi(-f_t^*)\right)} \cdot \int e^{-\frac{f_t^{*2}}{2}} \cdot \frac{1}{\sqrt{2\pi}}(-1)\frac{d}{dt} f_t^* d\gamma\right].$$

Recall $\frac{d}{dt} f_t^* = -g(\nabla f_t^*)$. So this becomes (after evaluating at $t = 0$).

$$= -\frac{I'\left(\int \Phi(-f_t^*)\right)}{I^2\left(\int \Phi(-f_t^*)\right)} \cdot \left(\int e^{-\frac{f^{*2}}{2}} \cdot \frac{1}{\sqrt{2\pi}} g(\nabla f^*) d\gamma\right)^2$$
$$+ \frac{1}{I\left(\int \Phi(-f_t^*)\right)} \cdot \left(\int -f^* e^{-\frac{f^{*2}}{2}} \cdot \frac{-1}{\sqrt{2\pi}} g(\nabla f^*)^2 d\gamma\right)$$
$$+ \frac{1}{I\left(\int \Phi(-f_t^*)\right)} \cdot \left(\int e^{-\frac{f^{*2}}{2}} \cdot \frac{-1}{\sqrt{2\pi}} \frac{d^2}{dt^2} f_t^* d\gamma\right).$$

We recall, by Lemma 2.2:

$$\frac{d^2}{dt^2} f_t^* = -\left\langle \nabla^2 f_{\nabla f^*} \nabla g(\nabla f), \nabla g(\nabla f)\right\rangle = -\left\langle \left(\nabla^2 W\right)^{-1} \nabla \phi, \nabla \phi\right\rangle,$$

where $\phi = g(\nabla f^*)$ and $W = f^*$. With this change of variables, and in view of the computation above, we see that (19) amounts to:

Theorem 3.10 (Barthe, Cordero-Erasquin, Ivanisvili, Livshyts [5])**.** *Consider convex* $W \geq 0$ *and consider the probability measure* $d\mu = e^{-\frac{W^2}{2}+C} d\gamma$*, let*

$$a = \int \Phi(-W) d\gamma \in [0,1],$$

and

$$A = \int e^{-\frac{W^2}{2}} d\gamma \cdot \Phi^{-1}(a) e^{\Phi^{-1}(a)^2/2}.$$

Then for any locally Lipschitz function h*,*

$$\int h^2 W d\mu - A\left(\int h d\mu\right)^2 \leq \int \left\langle \left(\nabla^2 W\right)^{-1} \nabla h, \nabla h\right\rangle d\mu.$$

Remark 3.12. *In fact, one could deduce Theorem 3.10 by linearizing Theorem 3.8, similar to how we deduced the Brascamp-Lieb inequality from the Generalized Log-Sobolev inequality; this is left as a home work. In fact, as we pointed out before, one could also deduce Brascamp-Lieb by taking the second derivative of Prekopa-Leindler inequality; this was also left as a home work.*

3.10 Home work

Question 3.1 (3 points)**.** *Find an alternative proof of Bobkov's inequality by approximating the Gaussian measure by the uniform measure on the Hamming cube.*

Question 3.2 (1 point)**.** *Verify that*

$$\frac{\phi^{-1}(\gamma(tB_2^n))}{t} \to_{t\to\infty} 1.$$

(recall that we used this fact to deduce the Gaussian Isoperimetric Inequality from Ehrhard's inequality).

Question 3.3 (1 point). *Deduce the Gaussian Isoperimetric Inequality directly from Bobkov's inequality.*

Question 3.4 (2 points). *Prove Kahane's inequality: let $g_1, ..., g_k, ...$ be a sequence of i.i.d. $N(0,1)$ random variables. For any $q \geq p > 0$, any $n \geq 1$ and any $z_1, ..., z_n \in \mathbb{R}^n$ we have*

$$\left(\mathbb{E}\|\sum_{i=1}^{n} g_i z_i\|^q\right)^{\frac{1}{q}} \leq \frac{\alpha_q}{\alpha_p}\left(\mathbb{E}\|\sum_{i=1}^{n} g_i z_i\|^p\right)^{\frac{1}{p}},$$

where

$$\alpha_p = (\mathbb{E}|g_i|^p)^{\frac{1}{p}}.$$

Question 3.5 (2 points). *Prove the following properties of Ehrhard symmetrizations. Let $S = S(L, e)$ be a Gaussian symmetrization and A and B be arbitrary closed sets. Then*

- *$\gamma(S(A)) = \gamma(A)$ provided that A is Borel measurable*
- *If $A \subset B$ then $S(A) \subset S(B)$*
- *For a vector v, $S(A + v) = S(A) + v$*
- *If $A_1 \subset A_2 \subset ...$ are open sets and $A = \cup_{i=1}^{\infty} A_i$ then $S(A) = \cup_{i=1}^{n} S(A_i)$*

Question 3.6 (1 point). *Let L_1 and L_2 be two sub-spaces in $\mathbb{R}^n$ such that $(L_1 \cap L_2)^{\perp} \cap L_1$ and $(L_1 \cap L_2)^{\perp} \cap L_2$ are orthogonal. Then*

$$S(L_1, e) \circ S(L_2, e) = S(L_2, e) \circ S(L_1, e) = S(L_1 \cap L_2, e).$$

Question 3.7 (1 point). *Let $n \geq 3$ and $k \geq 2$. Show that for every $k-$symmetrization S there exist 2-symmetrizations $S_1, ..., S_{k-1}$ such that $S = S_1 \circ ... \circ S_{k=1}$. Hint: use Question 3.6.*

Question 3.8 (1 point). *In dimension 2, show that there is a sequence $\theta_1, ..., \theta_k, ... \in \mathbb{S}^{n-1}$ such that letting $S_i = S(\theta_i^{\perp}, \theta_i) \circ ... \circ S(\theta_1^{\perp}, \theta_1)$, one has for every set A, that $S_i(A)$ converges to a half-space of the same Gaussian measure as A.*

Question 3.9 (2 points). *Prove, for any $\epsilon > 0$, any Gaussian symmetrization S and any set A :*

$$S(A) + \epsilon B_2^n \subset S(A + \epsilon B_2^n).$$

Conclude that the Ehrhard symmetrization decreases the Gaussian Perimeter. Using Questions 3.8 and 3.7, conclude the Gaussian Isoperimetric Inequality (directly without passing via the Ehrhard inequality).

Question 3.10 (2 points). *Prove that the Gaussian symmetrization of any convex set is also convex. (recall that this was a crucial step in proving Ehrhard's inequality.)*

Question 3.11 (2 points). *Find lower estimates on the isoperimetric profile of some product measures of your choice (beyond the uniform measure on the cube and the Gaussian).*

Question 3.12 (4 points). *Solve the isoperimetric problem on the square in dimension 2: prove that if $|A \cap [0,1]^2| = a \in [0,1]$ then $|\partial A \cap [0,1]^2|$ is bounded from below by the case of A being either an appropriately shifted ball, or a half-space.*

Question 3.13 (3 points). *Let L be a convex body. Find a lower estimate for the anisotropic Gaussian perimeter of a set A with $\gamma(A) = a$, that is*

$$\liminf_{\epsilon \to 0} \frac{\gamma(A + \epsilon L) - \gamma(A)}{\epsilon}.$$

For which L is it sharp?

Question 3.14 (2 points). *Prove the simple case of the Gaussian Correlation Inequality (called the Sidak Lemma): let K and L be a pair of symmetric strips. Then $\gamma(K \cap L) \geq \gamma(K)\gamma(L)$.*

Hint: use the Prekopa-Leindler inequality.

Question 3.15 (1 point). *Prove the Gaussian Log-Sobolev inequality by linearizing Bobkov's inequality.*

Question 3.16 (1 point). *Show that the functional Ehrhard inequality tensorizes, i.e. that from knowing it in dimensions k and m one can deduce it in the dimension $k + m$.*

Question 3.17 (5 points). *Try and find the proof of Functional Ehrhard Inequality in dimension one, without using the geometric Ehrhard.*

Question 3.18 (1 point). *Verify that for $a \in [0,1]$,*

$$\eta(a) = \sqrt{2\pi} a \Phi^{-1}(a) e^{\Phi^{-1}(a)^2/2} \geq -1. \tag{20}$$

Question 3.19 (3 points). *In class we showed that if K is any convex set, $\gamma(K) = a \in [0,1]$, then letting $\eta(a)$ as in (20) we have*

$$\frac{1}{\gamma(K)} \int_K \langle x, \theta \rangle^2 \, d\gamma + \frac{\eta(a)}{\gamma(K)^2} \left(\int_K \langle x, \theta \rangle \, d\gamma \right)^2 \leq 1.$$

Find an alternative proof of this fact using Ehrhard's inequality, or perhaps the consequences of Ehrhard's inequality – the generalized Bobkov inequality or the Ehrhard-Brascamp-Lieb inequality which we deduced in class.

4 The Blaschke-Santaló inequality

4.1 Steiner symmetrization

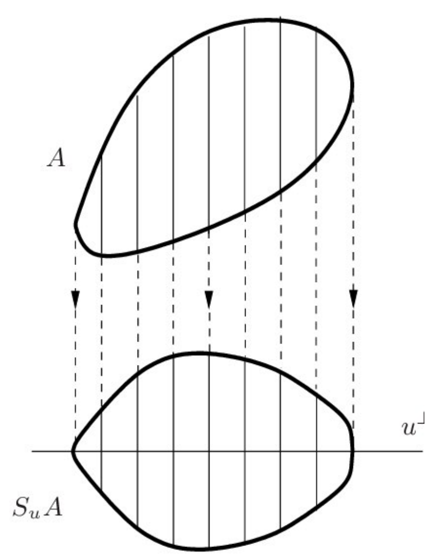
Steiner symmetrization is a technique which was invented by Jacob Steiner to prove the isoperimetric inequality in 1837. Given a hyperplane $\theta^\perp$ and a set K, take every one-dimensional section of K parallel to θ and replace it with the interval symmetric about $\theta^\perp$, of the same length. Then take a union of all these intervals, and get the Steiner symmetral of K about $\theta^\perp$. This new set, which is denoted $S_\theta(K)$, has the same volume as K (as we shall explain below), and it has in many ways better "isoperimetric properties" than K. In modern geometry this technique is used for various purposes, and we will see several uses of it in this course. We now proceed with a formal definition.

Definition 4.1 (Steiner symmetrization). Let $\theta \in \mathbb{S}^{n-1}$, and $K \subseteq \mathbb{R}^n$ Borel-measurable. The Steiner symmetrization of K is

$$S_\theta(K) = \bigcup_{y \in \theta^\perp} S_\theta(K \cap \{y + t\theta \colon t \in \mathbb{R}\})$$

where $S_\theta(K \cap \{y+t\theta \colon t \in \mathbb{R}\})$ is the interval symmetric about $\theta^\perp$, contained in $\{y+t\theta \colon t \in \mathbb{R}\}$, and of length equal to the Lebesgue measure of the set $K \cap \{y + t\theta \colon t \in \mathbb{R}\}$.

Remark 4.1. *Note that Borel-measurability of K implies that for every $y \in \theta^\perp$, the set $K \cap \{y + t\theta \colon t \in \mathbb{R}\}$ is measurable.*



We observe the key property of Steiner symmetrization – the fact that it preserves volume:

Claim 4.1. $|K| = |S_\theta(K)|$.

Proof. Using Fubini's theorem, we write

$$|K| = \int_{\theta^\perp} \int_{-\infty}^{\infty} |K \cap \{y + t\theta \colon t \in \mathbb{R}\}|_1 dt = \int_{\theta^\perp} \int_{-\infty}^{\infty} |S_\theta(K) \cap \{y + t\theta \colon t \in \mathbb{R}\}|_1 dt = |S_\theta(K)|.$$

□

Now we define a notion of Hausdorff distance between convex bodies.

Definition 4.2 (Hausdorff distance)**.** The Hausdorff distance between convex bodies $K, L \subseteq \mathbb{R}^n$ is defined as

$$d_H(K, L) = \inf\{t > 0 \colon \exists \alpha > 0 \text{ s.t. } K \subseteq \alpha L \subseteq t\alpha K\}.$$

Properties of the Steiner symmetrization

- $S_\theta(K)$ is convex whenever K is convex.
- $\mathrm{circ}(K) \geq \mathrm{circ}(S_\theta(K))$ where the circum-radius is defined as

$$\mathrm{circ}(L) = \inf\{s > 0 \colon \exists y \in \mathbb{R}^n, L \subseteq sB_2^n + y\}.$$

- $\mathrm{inrad}(K) \leq \mathrm{inrad}(S_\theta(K))$ where the in-radius is defined as

$$\mathrm{inrad}(K) = \sup\{t > 0 \colon \exists y \in \mathbb{R}^n, tB_2^n + y \subseteq K\}.$$

- $\lambda S_\theta(K) = S_\theta(\lambda K)$ for all $\lambda \geq 0$.
- S_θ is continuous in the Hausdorff distance.
- $S_\theta(K) + S_\theta(L) \subseteq S_\theta(K + L)$.
- $|\partial S_\theta(K)|_{n-1} \leq |\partial K|_{n-1}$.
- $\mathrm{diam}(K) \geq \mathrm{diam}(S_\theta(K))$, where the diameter of a set A is

$$diam(A) = \sup_{x,y \in A} |x - y|.$$

We leave these properties as a home work.

At last, we prove another very important fact about Steiner symmetrization: successive Steiner symmetrizations of a given set converge to the Euclidean ball (of the same volume).

Claim 4.2. *There exists a sequence $\{\theta_k\} \subset \mathbb{S}^{n-1}$ such that for all convex bodies K,*

$$K, S_{\theta_1}(K), S_{\theta_2}(S_{\theta_1}(K)), ... \to RB_2^n$$

where $R = \frac{|K|^{1/n}}{|B_2^n|^{1/n}}$, and the convergence is in the Hausdorff distance.

Sketch proof. Since K is compact, there exists $t > 0$ such that $K \subset tB_2^n$. Consider the family Ω of all successive Steiner symmetrals of K. Note that by our properties, all of these symmetrals are also contained in tB_2^n. Let $r = \inf_{L\in\Omega} \operatorname{circ}(L)$ (where once again $\operatorname{circ}(L)$ stands for the circum-radius), and consider a sequence of radii $r_k \to r$, with the corresponding bodies Q_k.

The Blaschke selection theorem (see e.g. [1]) states that for any family of convex bodies contained in tB_2^n, there exists a convergent sub-sequence. Since $\operatorname{circ}(K) \geq \operatorname{circ}(S_\theta(K))$, there exists a sequence $\{L_k\} \subseteq \Omega$ such that L_k converges to L with $\operatorname{circ}(L) = r > 0$.

We claim that L is a ball. Indeed, suppose not. Then L misses a cap of the ball rB_2^n. By compactness we may cover the boundary of the ball rB_2^n with rotations of this cap, corresponding to directions $\theta_1, ..., \theta_m$. Then, symmetrizing L with respect to $\theta_1, ..., \theta_m$, we get a body with a strictly smaller in-radius, which contradicts our choice of L. □

4.2 The formulation of the Blaschke-Santaló inequality

Let K be a symmetric convex body, recall

$$K^\circ = \{x \colon \forall y \in K, \langle x, y\rangle \leq 1\}$$

is its polar. Some notable examples include $(B_2^n)^o = B_2^n$ and, more generally, $(B_p^n)^o = B_q^n$ where $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$

Let T a linear operator. Recall that $(TK)^\circ = (T^{-1})^\top K^\circ$. The volume product $|K| \cdot |K^\circ|$ is affine invariant:

$$|TK| \cdot |(TK)^\circ| = \det T|K| \det T^{-1}|K^\circ| = 1 \cdot |K| \cdot |K^\circ| = |K| \cdot |K^\circ|$$

In particular, for any ellipsoid E,

$$|E| \cdot |E^\circ| = |B_2^n|^2 \sim \left(\frac{2\pi e^2}{n}\right)^n,$$

and for any parallelpiped P,

$$|P| \cdot |P^\circ| = |B_\infty^n| \cdot |B_1^n| = \frac{4^n}{n!} \sim \frac{(4e)^n}{n^n}.$$

We formulate the celebrated

Theorem 4.1 (Blaschke-Santalo inequality [30])**.** *For any symmetric convex body K,*

$$|K| \cdot |K^\circ| \leq |B_2^n|^2$$

And what about the estimate from below?

Conjecture 4.1 (Mahler, 1937 (symmetric version))**.** *For a symmetric convex body K in $\mathbb{R}^n$, $|K| \cdot |K^\circ| \geq \frac{4^n}{n!} = |B_\infty^n| \cdot |B_1^n|$.*

Mahler proved it in dimension 2, but also see home work. Iryeh, Shibata proved it in dimension 3, and their proof was later simplified by Fradelizi, Hubard, Meyer, Roldan-Pensado, Zvavitch.

4.3 Proof of the symmetric Blaschke-Santalo inequality

The proof uses Steiner symmetrization. We shall show:

Lemma 4.1. *$|K^\circ| \le |S_u(K)^\circ|$ for all $u \in \mathbb{S}^{n-1}$.*

Proof. WLOG, suppose $u = e_n$. We write

$$S_{e_n}(K) = \{(x, \frac{s-t}{2})\colon (x,s),(x,t) \in K\}$$

Therefore,

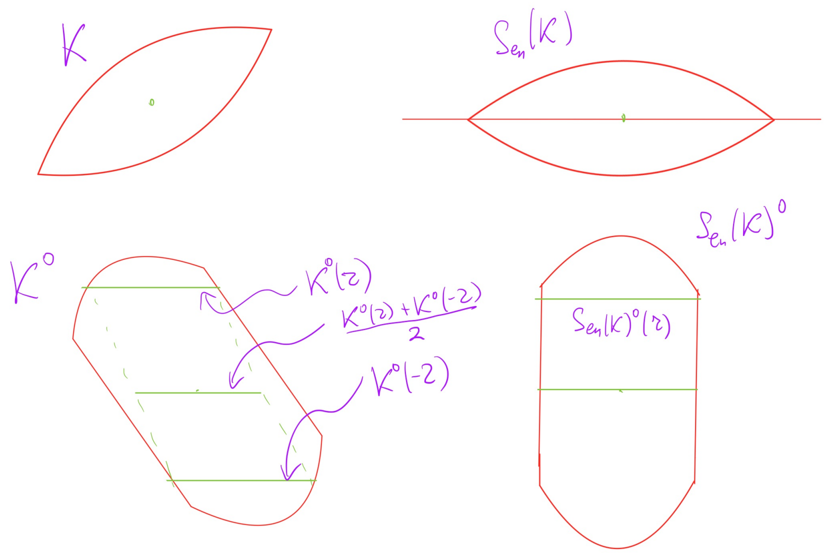
$$S_{e_n}(K)^\circ = \{(y,z)\colon \langle x,y\rangle + z\frac{s-t}{2} \le 1 \ \forall (x,s),(x,t) \in K\}$$

Define the slice $L(r) = \{x \in \mathbb{R}^{n-1}\colon (x,r) \in K\}$ for $L \subseteq \mathbb{R}^n$. Consider

$$\frac{K^\circ(r) + K^\circ(-r)}{2} = \{\frac{y+z}{2}\colon \langle x,y\rangle + sr \le 1, \langle w,z\rangle - tr \le 1,\ \forall (x,s),(w,t) \in K\}.$$

By reducing the number of restrictions, we obtain

$$\begin{aligned}\frac{K^\circ(r) + K^\circ(-r)}{2} &\subseteq \{\frac{y+z}{2}\colon \langle x,y\rangle + sr \le 1, \langle x,z\rangle - tr \le 1,\ \forall (x,s),(x,t) \in K\} \\ &\subseteq \{\frac{y+z}{2}\colon \langle x, \frac{y+z}{2}\rangle + \frac{s-t}{2} r \le 1 \ \forall (x,s),(x,t) \in K\} \\ &= \{v\colon \langle x,v\rangle + \frac{s-t}{2} r \le 1 \ \forall (x,s),(x,t) \in K\} \\ &= S_{e_n}(K)^\circ(r).\end{aligned}$$



Next, $|K(r)|$ is an even function of r because K is symmetric. By the Brunn-Minkowski inequality,

$$\left|\frac{K^\circ(r)+K^\circ(-r)}{2}\right| \geq \sqrt{|K^\circ(r)\cdot|K^\circ(-r)|} = |K^\circ(r)|,$$

which implies $|S_{e_n}(K)^\circ(r)| \geq |K^\circ(r)|$. Therefore, using Fubini's theorem, we get

$$|K^\circ| = \int_{-\infty}^{\infty} |K^\circ(r)|\,\mathrm{d}r \leq \int_{-\infty}^{\infty} |S_{e_n}(K)^\circ(r)|\,\mathrm{d}r = |S_{e_n}(K)^\circ|.$$

□

In order to derive the Blaschke-Santalo inequality from Lemma 4.1, we select a sequence of directions such that the successive symmetrizations of K approach a ball, and note that the polar volume increases along this sequence, while the volume remains preseved. Namely, choose a sequence $u_1, u_2, \dots$ such that $S_{u_k,\dots,u_1}K \to RB_2^n$, where $R = \frac{|K|^{1/n}}{|B_2^n|^{1/n}}$. Then

$$|K|\cdot|K^\circ| \leq |RB_2^n||(RB_2^n)^\circ| = |B_2^n|^2.\square$$

4.4 Functional version of the Blaschke-Santalo inequality

Below we present a functional version of the Blaschke-Santalo inequality. Introduced by Ball [3]. We shall see that it implies the usual (geometric) Blaschke-Santalo inequality; our proof will also be based on the geometric version, following the work of Arstein-Avidan, Klartag, Milman [2]. For simplicity, we focus on the symmetric version, but a non-symmetric functional Blaschke-Santalo is available too [2]. We recommend also the proof by Lehec [25] which did not rely on the geometric Blaschke-Santalo inequality.

Theorem 4.2 (Ball [3]; Arstein-Avidan, Klartag, Milman [2]; Lehec [25]). *If $\psi : \mathbb{R}^n \to \mathbb{R}$ is an even function such that $\int e^{-\psi} < \infty$, then*

$$\int e^{-\psi}\cdot\int e^{-\psi^\star} \leq \left(e^{-x^2/2}\right)^2 = (2\pi)^n.$$

Recall that given a function ψ, the function ψ^* is the smallest of the functions φ which satisfy for every $x, y \in \mathbb{R}^n$ the inequality $\psi(x) + \varphi(y) \geq \langle x, y\rangle$. Therefore, we get:

Corollary 4.1. *Suppose that $f, g : \mathbb{R}^n \to \mathbb{R}$ are such that $f(x)\cdot g(y) \leq e^{-\langle x,y\rangle}$, then*

$$\int f \cdot \int g \leq (2\pi)^n.$$

Proof of theorem 4.2. For any constant c, we have $(\psi + c)^\star = \psi^\star - c$, and so we can assume that $\psi \geq 0$. Moreover, we assume that $\psi(0) = 0$ which implies that $\psi^\star(0) = 0$ and $\psi^\star \geq 0$.

Furthermore, we can assume without loss of generality that ψ is convex. Indeed, otherwise we can replace the left hand side with ψ^{**} and it only increases.

We have

$$\int_{\mathbb{R}^n} e^{-\psi}\,\mathrm{d}x = \int_0^\infty |\{e^{-\psi} > t\}|\,\mathrm{d}t$$
$$= \int_0^\infty e^{-s}|\{\psi < s\}|\,\mathrm{d}s,$$

where the first equality follows by the Fubini theorem, the second equality applies the change of variable $t = e^{-s}$. Similarly, we obtain

$$\int_{\mathbb{R}^n} e^{-\psi^\star}\,\mathrm{d}x = \int_0^\infty e^{-s}|\{\psi^\star < s\}|\,\mathrm{d}s.$$

We make the following claim.

Claim 4.3. *For any $s, t \geq 0$,*

$$\{\psi^\star < t\} \subset (s+t)\cdot\{\psi < s\}^\circ.$$

Proof. Consider $x \in \{\psi < s\}$ and $y \in \{\psi^\star < t\}$ and note that it suffices to show that $\langle x, y\rangle \leq (s+t)$. By the property of the Legendre transform, we know that

$$\langle x, y\rangle \leq \psi(x) + \psi^\star(y) \leq s + t.$$

□

Consider the following three functions on $\mathbb{R}_+$:

- $f(s) = e^{-s}\cdot|\{\psi < s\}|$
- $g(t) = e^{-t}\cdot|\{\psi^\star < t\}|$
- $h(x) = |B_2^n|\cdot 2^{n/2}\cdot e^{-x}\cdot x^{n/2}$

We will apply the Prekopa-Leindler inequality on these functions. First, we claim that

$$h\left(\frac{s+t}{2}\right) \geq \sqrt{f(s)\cdot g(t)}.$$

Indeed, we can write

$$h^2\left(\frac{s+t}{2}\right) = |B_2^n|^2\cdot 2^n\cdot e^{-(s+t))}\cdot\left(\frac{s+t}{2}\right)^n = |B_2^n|^2\cdot e^{-s}\cdot e^{-t}\cdot(s+t)^n.$$

We have,

$$\begin{aligned} f(s)\cdot g(t) &= e^{-s}\cdot|\{\psi < s\}|\cdot e^{-t}\cdot|\{\psi^\star < t\}| \\ &\leq e^{-s}\cdot|\{\psi < s\}|\cdot e^{-t}\cdot(s+t)^n\cdot|\{\psi < s\}^\circ| \\ &\leq e^{-s}\cdot e^{-t}\cdot(s+t)^n\cdot|B_2^n|^2 = h^2\left(\frac{s+t}{2}\right), \end{aligned}$$

where the first inequality uses Claim 4.3, and the second inequality uses the Blaschke-Santalo inequality. Thus, the three functions satisfy the conditions of the Prekopa-Leindler inequality, and we get

$$\int e^{-\psi} \cdot \int e^{-\psi^\star} = \int e^{-s}|\psi < s| \cdot \int e^{-t}|\psi^\star < t| \leq \left(\int e^{-x} \cdot x^{n/2}\right)^2 \cdot 2^n \cdot |B_2^n|^2 = (2\pi)^n.$$

$\square$

Remark 4.2. *Setting $\psi(x) = \frac{||x||_K^2}{2}$ and $\psi^\star(x) = \frac{||x||_{K^\circ}^2}{2}$, and integrating in polar coordinates recovers the usual Blaschke-Santalo inequality (see home work).*

Remark 4.3. *The equality case occurs if and only if $\psi(x) = \frac{|x|^2}{2}$, as was shown in [2].*

We also mention the following theorem (see HW).

Theorem 4.3 (Fradelizi-Meyer [16])**.** *Consider even functions $f, g : \mathbb{R}^n \to \mathbb{R}$ such that for all $x, y, \in \mathbb{R}^n$, $f(x) \cdot g(y) \leq \rho(\langle x, y\rangle)$ whenever $\langle x, y\rangle \geq 0$. Then,*

$$\left(\int f\right) \cdot \left(\int g\right) \leq \left(\int \rho\left(|x|^2\right)\right)^2.$$

This holds for any $\rho : \mathbb{R}_+ \to \mathbb{R}_+$.

4.5 Linearizing Theorem 4.2.

Take $\psi = \frac{|x|^2}{2} + \epsilon f$ for some function f. Then, we have

$$\int e^{-\left(\frac{|x|^2}{2}+\epsilon f\right)} \cdot \int e^{-\left(\frac{|x|^2}{2}+\epsilon f\right)^\star} \leq (2\pi)^n.$$

Recall that

$$v_t^\star = v_t - t\dot{v}_t(\nabla v_t) - \frac{t^2}{2}\ddot{v}_t(\nabla v_t) + \frac{t^2}{2}\langle(\nabla^2 v_t)^{-1}\nabla[\dot{v}_t|_{\nabla v_t}], \nabla[\dot{v}_t|_{\nabla v_t^*(x)}]\rangle.$$

Then

$$\left(\frac{|x|^2}{2} + \epsilon f\right)^\star = \frac{|x|^2}{2} - \epsilon f + \frac{\epsilon^2}{2}|\nabla f|^2 + o(\epsilon^2),$$

since $\nabla v_0 = x$ and $\nabla^2 v_0 = \mathrm{Id}$. So, we have (up to the terms of order $o(\epsilon^2)$):

$$\int e^{-\frac{|x|^2}{2}-\epsilon f} \cdot \int e^{-\frac{|x|^2}{2}+\epsilon f-\frac{\epsilon^2}{2}|\nabla f|^2} \leq (2\pi)^n.$$

Using $e^{-\delta} = 1 - \delta + \frac{\delta^2}{2}$ (up to lower order terms), we get

$$(2\pi)^n \geq \left(\int e^{-\frac{|x|^2}{2}} \cdot \left(1 - \epsilon f + \frac{\epsilon^2}{2}f^2\right)\right) \cdot \left(\int e^{-\frac{|x|^2}{2}} \cdot \left(1 - \epsilon f + \frac{\epsilon^2}{2}|\nabla f|^2 - \frac{\epsilon^2 f^2}{2}\right)\right) + o(\epsilon^2).$$

Dividing both sides by $(2\pi)^n$ gives

$$1 \geq \left(\int \left(1 - \epsilon f + \frac{\epsilon^2}{2} f^2\right) \mathrm{d}\gamma\right) \cdot \left(\int \left(1 - \epsilon f + \frac{\epsilon^2}{2}|\nabla f|^2 - \frac{\epsilon^2 f^2}{2}\right) \mathrm{d}\gamma\right) + o(\epsilon^2).$$

where $\mathrm{d}\gamma$ is the Gaussian measure. We note that the constant terms cancel out, and so do the terms which are multiplied by ϵ. Collecting the terms multiplied by ϵ^2 gives the following inequality:

Theorem 4.4. *For all even functions f, we have*

$$\int_{\mathbb{R}^n} f^2 \,\mathrm{d}\gamma - \left(\int_{\mathbb{R}^n} f \,\mathrm{d}\gamma\right)^2 \leq \frac{1}{2} \int_{\mathbb{R}^n} |\nabla f|^2 \,\mathrm{d}\gamma.$$

Remark 4.4. *Note the improved the constant as compared to the Gaussian Poincare's inequality which we obtained from Theorem 2.6.*

Remark 4.5. *We note that using the non-symmetric version of the Blaschke-Santalo inequality, it suffices to assume $\int \nabla f \,\mathrm{d}\gamma = 0$, as integrating by parts, we obtain*

$$\int \frac{\partial f}{\partial x_i} \,\mathrm{d}\gamma = \int \langle \nabla f, \nabla x_i \rangle \,\mathrm{d}\gamma = -\int f \cdot L_\gamma x_i \,\mathrm{d}\gamma = \int f \cdot x_i \,\mathrm{d}\gamma.$$

So, $\int \nabla f \,\mathrm{d}\gamma = 0$ is equivalent to showing that for all linear functions $\langle x, \theta \rangle$, we have $\int f \cdot \langle x, \theta \rangle \,\mathrm{d}\gamma = 0$. This implies that the second eigenvalue of the Ornstein–Uhlenbeck operator is 2.

Remark 4.6. *The same result could be obtained using methods of Fourier Analysis and the decomposition into Hermite polynomials.*

4.6 Blaschke-Santaló type inequality with non-round extremizers

Theorem 4.5 (Colesanti, Kolesnikov, Livshyts, Rotem)**.** *Let $p > 1$ and let V be an even strictly convex p-homogeneous C^2 function on $\mathbb{R}^n$.*

Assume that V is an unconditional function such that for every $x \in \mathbb{R}^n$ with non-negative coordinates, the function $V\left(x_1^{\frac{1}{p}}, ..., x_n^{\frac{1}{p}}\right)$ is concave in x. Then inequality

$$\int e^{-\Phi(x)} dx \left(\int e^{-\frac{1}{p-1}\Phi^*(\nabla V(y))} dy\right)^{p-1} \leq \left(\int e^{-V(x)} dx\right)^p \tag{21}$$

as well as the inequality

$$\mathrm{Var}_\mu f \leq \left(1 - \frac{1}{p}\right) \int \langle (D^2 V)^{-1} \nabla f, \nabla f \rangle d\mu. \tag{22}$$

hold for every unconditional convex Φ.

Assume, in addition, that for every coordinate hyperplane H, with unit normal e, for every $x' \in H$, the function $\varphi\colon [0,+\infty) \to \mathbb{R}$ defined by

$$\varphi(t) = \det D^2 V^*(x' + te)$$

is decreasing. Then the same pair of inequalities hold for every even convex Φ.

Corollary 4.2. *Let $V = c|x|_q^p$. Then inequality (21) and (22) holds in the following cases:*

1. *For $p \geq q > 1$ and unconditional Φ*
2. *For $p \geq q \geq 2$ and even Φ.*

Proposition 4.1. *Let $p > 1$ and let V be an even strictly convex p-homogeneous C^2 function on $\mathbb{R}^n$. Inequality (21) holds for arbitrary convex proper function Φ if and only if inequality*

$$|K| \cdot |\nabla V^*(K^o)|^{p-1} \leq \left| \left\{ V \leq \frac{1}{p} \right\} \right|^p \tag{23}$$

holds for arbitrary compact convex body K.

If inequality (23) holds, the equality is attained when K is a level set of V : $K = \{V \leq \alpha\}$.

We conclude with an unusual isoperimetric-type inequality in which the minimizers are not round.

Corollary 4.3. *Suppose $p \geq 2$. Let K be a symmetric convex body in $\mathbb{R}^n$, $n \geq 2$. Then*

$$\left(|K| \left(\int_{K^\circ} \prod_{i=1}^n |x_i|^{\frac{2-p}{p-1}} dx \right)^{p-1} \right) \leq |B_p^n|^p,$$

with equality when $K = B_p^n$.

4.7 Home work

Question 4.1 (1 point)**.** *Let P be a polytope given by*

$$P = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq a_i, \, \forall i = 1, ..., N\},$$

for some unit vectors $u_1, ..., u_N$ and positive numbers $a_1, ..., a_N$, and suppose that P is bounded. Show that

$$P^o = \bar{conv} \left\{ \frac{u_1}{a_1}, ..., \frac{u_N}{a_N} \right\}.$$

Conclude that $(B_1^n)^o = B_\infty^n$.

Question 4.2 (1 point). *In this question, K and L stand for convex bodies in $\mathbb{R}^n$ with non-empty interior, containing the origin.*

a) Prove that $K^{oo} = K$.

b) Prove that for a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ with $\det T \neq 0$,

$$(T^t K)^o = T^{-1} K^o.$$

Conclude that a polar of an ellipsoid is an ellipsoid.

c) Prove that

$$(B_p^n)^o = B_q^n,$$

where $\frac{1}{p} + \frac{1}{q} = 1$, for all $p, q > 1$.

d) Prove that

$$(K \cap L)^o = conv(K^o \cup L^o).$$

e) Prove that for every subspace H of $\mathbb{R}^n$

$$(K|H)^o \cap H = K^o \cap H.$$

f) Prove that if $K \subset L$, one has $L^o \subset K^o$.

g) Prove that if K is symmetric then K^o is symmetric.

h) Show that for any (possibly non-convex) set A, we have $A^o = (conv(A))^o$. Conclude that the polar is always a convex set.

Question 4.3 (1 point). *Let K be a symmetric convex body. Show that if $K = K^o$ then $K = B_2^n$.*

Question 4.4 (1 point). *Show that for any symmetric convex body K, we have*

$$h_K(\theta)\rho_{K^o}(\theta) = 1$$

for all $\theta \in \mathbb{R}^n$.

Question 4.5 (3 points). *Verify Mahler's conjecture in $\mathbb{R}^2$ for symmetric polygons: show that for any symmetric polygon P in $\mathbb{R}^2$,*

$$|P| \cdot |P^o| \geq 8.$$

Question 4.6 (1 point). *Given a Borel measurable set A in $\mathbb{R}^n$, a function $\alpha : A \to \mathbb{R}$ and a vector $v \in \mathbb{R}^n \setminus 0$, consider the* shadow system

$$K_t = conv\{x + \alpha(x)v \, : x \in A\},$$

and define the convex body

$$\tilde{K} = conv\{x + t\alpha(x)e_{n+1}\} \subset \mathbb{R}^{n+1}.$$

Show that for $u \in e_{n+1}^{\perp}$,

$$h_{K_t}(u) = h_{\tilde{K}}(u + t\langle u, v\rangle e_{n+1}).$$

Question 4.7 (2 points). *Prove the Blaschke-Santalo inequality using shadow systems.*
Hint 1. Express $|K_t^o|$ combining the formulas from Questions 4.4 and 4.6.
Hint 2. pass the integration on $\mathbb{S}^{n-1}$ to the integration on $B_2^{n-1} = \{x \in \mathbb{R}^n : \langle x, v\rangle = 0\}$ with the map $x = \theta - \langle \theta, v\rangle v$.
Hint 3: now extend the integration to $\mathbb{R}^{n-1}$.
Hint 4. Conclude that $|K_t^o|$ is -1-concave in t for any shadow system, using Question 2.9.
Hint 5. Notice that Steiner symmetrization can be realized as a shadow system, and, using the fact that $|K^o| = |\bar{K}^o|$ for any reflection $\bar{K}$ of K, and the $-1-$concavity of $|K_t^o|$ along any shadow system, conclude that Steiner symmetrization increases $|K^o|$. Conclude the Blaschke-Santalo inequality.
(this proof was discovered by Campi and Gronchi).

Question 4.8 (1 point). *a) For any $\varphi : \mathbb{R} \to \bar{R}$ one has φ^* is a convex function.*
b) If φ is convex then $\varphi^{**} = \varphi$.
c) If $f \geq g$ then $f^* \leq g^*$.
d) Find $|x_1|^*$.
e) Find $(\frac{\|x\|_p^q}{q})^*$.
f) For a convex body K, one has $(-\log 1_K)^* = h_K$.
g) For an $a \in \mathbb{R}$, find $(a\varphi)^*$ in terms of φ^*.
h) Letting $\varphi_a(x) = \varphi(ax)$ for some $a \in \mathbb{R}$, find φ_a^*.
i) Show that $(\varphi + a)^* = \varphi^* - a$, for any $a \in \mathbb{R}$.
j) Show that

$$(f^* + g^*)^*(z) = \inf_{x,y \in \mathbb{R}^n : x+y=z} (f(x) + g(y)) .$$

k) Fix $\alpha > 1$. Show that is f is $\alpha-$homogeneous (i.e. $f(tx) = t^\alpha f(x)$ for all $t \in \mathbb{R}$) then $f^*(\nabla f) = (\alpha - 1)f$.
Hint: use one of the properties we proved in class, combined with the fact that for an $\alpha-$homogeneous function one has $\langle \nabla f, x\rangle = \alpha f$ (verify this).

Question 4.9 (1 point). *Find an alternative short proof of the functional Blaschke-Santalo inequality for unconditional functions by passing the integration from $\mathbb{R}^n$ to the set*

$$\{x \in \mathbb{R}^n : \forall i = 1, ..., n,\ x_i \geq 0\},$$

and making a change of variables in the Prekopa-Leindler inequality given by $(x_1, ..., x_n) = (e^{t_1}, ..., e^{t_n})$. (see also a similar Question 2.10).

Question 4.10 (1 point). *Show that the Santaló point of a convex body exists and is unique.*

Question 4.11 (4 points). *Find a statement and a proof for the Blaschke-Santalo inequality and the functional Blaschke-Santalo inequality for non-symmetric convex sets and non-even functions (as per our discussion in class).*

Question 4.12 (3 points). *a) Note that the Blaschke-Santalo inequality on the plane is equivalent to showing that for any even periodic function $h \in C^2([-\pi,\pi])$, such that $h \geq 0$ and $h + \ddot{h} \geq 0$,*

$$F(h) = \int_{-\pi}^{\pi} h^{-2} dt \cdot \int_{-\pi}^{\pi} h^2 - \dot{h}^2 dt \leq 4\pi^2.$$

(or equivalently, one may drop the even assumption and restrict to $[0,\pi]$).

Hint: use Question 4.4 to conclude that

$$|K^o| = \frac{1}{2} \int_{-\pi}^{\pi} h^{-2} dt.$$

Also use Question 2.23.

b) Observe that the equality is attained when h is the support function of an ellipse.

c) Find some way to show that this inequality is true.

Option 1: maybe use basic Harmonic Analysis (I don't know if it is possible and would love to see it if it works)?

Option 2: maybe use variational approach? That is, suppose that a given function h maximizes the functional $F(h)$, argue that it suffices to assume that $h \in C^1([-\pi,\pi])$ and $h > 0$ and $h + \dot{h} > 0$, then argue that for any $\epsilon > 0$ and any even smooth $\psi > 0$, $\frac{d}{d\epsilon} F(h + \epsilon\psi) = 0$, and conclude some ODE that h must satisfy (in view of the arbitrarity of ψ). Then conclude that the support function of an ellipsoid is the only type of function that satisfies this ODE.*

** This "argue" may not be easy and you are encouraged to pursue other steps in this hint even if this step is not clear at first.*

Option 3: try whatever you like! :)

Question 4.13 (5 points). *a) Find an example of a non-symmetric convex body for which the Santaló point and the center of mass do not coincide.*
b) How far could they be?
c) For a convex body K in $\mathbb{R}^n$, let $d(K)$ be the distance between the center of mass and the Santaló point. How large could $\frac{d(K)}{diam(K)}$ be?

Question 4.14 (1 point). *Let H be a Hanner polytope (as defined inductively in class). Show that indeed*

$$|H||H^o| = \frac{4^n}{n!}.$$

Question 4.15 (2 points, Saint-Raimond's theorem via Meyer's proof). *Prove the (symmetric) Mahler conjecture in the case when the body K is unconditional (that is, it is symmetric with respect to every coordinate hyperplane).*
Hint 1: Note that the result is true in dimension 1 and proceed by induction.
Hint 2: Consider $K^+ = \{x \in K : x_i \geq 0 \ \forall i = 1, ..., n\}$. Given a point $x \in K^+$ consider n cones

$$K_i = conv\{x, K^+ \cap e_i^{\perp}\}.$$

Note that

$$|K| \geq 2^n \sum_{i=1}^{n} |K_i|,$$

write the above out to deduce that the vector with coordinates $(..., \frac{2|K \cap e_i^{\perp}|}{n|K|}, ...)$ *belongs to* K^o *(use the unconditionality in the process).*
Hint 3: Do the same argument for K^o*, and then use properties of polarity along with the fact that* $K \cap e_i^{\perp} = K|e_i^{\perp}$ *(which is another place where the fact that* K *is unconditional is used!!!), to conclude that*

$$|K||K^o| \geq \frac{4}{n^2} \sum_{i=1}^{n} |K \cap e_i^{\perp}| \cdot |(K \cap e_i^{\perp})^o|,$$

and use induction.

Question 4.16 (5 points). *Iryeh and Shibata's proof of Mahler's conjecture in* $\mathbb{R}^3$ *followed the same idea as in Question 4.15, and hinged on the fact that it is possible to bring a symmetric convex body in* $\mathbb{R}^3$ *into a position where it is possible to split it into* 8 *parts with coordinate hyperplanes so that each part has the same volume, and each of the three coordinate hyperplane sections of* K *is split into four equal parts, and also each projection of* K *onto coordinate hyperplane coincides with a section.*
a) verify that this fact ensures the validity of Mahler conjecture (in the same way as above);
b) prove this challenging fact.

Question 4.17 (3 points). *Verify the non-symmetric Mahler conjecture in dimension* 2.

Question 4.18 (3 points). *Using the ideas from Question 4.15, prove the result of Barthe, Fradelizi: if a convex body* K *in* $\mathbb{R}^n$ *has all the symmetries of the regular simplex then it verifies the non-symmetric Mahler conjecture, that is,* $|K||K^o| \geq |S_n|^2$ *where* S_n *is the self-dual regular simplex.*

Question 4.19 (10 points). *Is it possible to use the ideas from Question 4.18 to prove the non-symmetric Mahler conjecture in* $\mathbb{R}^3$*, that is, to show that for any convex body* K *in* $\mathbb{R}^3$ *one has* $|K||K^o| \geq |S_3|^2$ *where* S_3 *is the self-dual regular simplex? Maybe one could prove the appropriate non-symmetric version of the fact proved by Iryeh and Shibata about bringing* K *into a certain position?*

Question 4.20 (2 points). *Prove the following result of Fradelizi and Meyer: Mahler's conjecture is equivalent to the following functional version. For any convex function* φ *on* $\mathbb{R}^n$ *one has*

$$\int e^{-\varphi} \cdot \int e^{-\varphi^*} \geq 4^n.$$

Question 4.21 (2 points). *Prove the following result of Fradelizi and Meyer which extends the functional Blaschke-Santalo: let* $\rho : [0, \infty) \to [0, \infty)$ *be a measurable function and suppose*

f and g are even log-concave functions such that $f(x)g(y) \leq \rho^2(\langle x, y\rangle)$ whenever $\langle x, y\rangle \geq 0$. Then

$$\int f \cdot \int g \leq \left(\int \rho(|x|^2)\right)^2.$$

Question 4.22 (5 points)**.** *We saw in class that the $p-$Beckner inequality on the circle for periodic functions*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2 - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f^p\right)^{\frac{2}{p}} \leq (2-p)\frac{1}{2\pi}\int_{-\pi}^{\pi} \dot{f}^2$$

holds not only for $p \in [1,2)$ but also for $p=-2$. By any chance, is it possible to argue that there is a range of negative p for which this holds (rather than just one value $p=-2$)? Maybe argue similarly to Question 2.33?

Question 4.23 (2 points)**.** *Show that Talagrand's transport-entropy inequality implies the Gaussian Poincare inequality.*
Hint: linearize.

Question 4.24 (10 points)**.** *Try and make some progress on the question we discussed in class: for any even log-concave measure μ and any symmetric convex body K one has*

$$\mu(K)\mu(K^o) \leq \mu(B_2^n)^2.$$

Maybe you can find a proof in some partial case – for some class of measures, for unconditional measures/bodies, in dimension 2, etc?

Question 4.25 (1 point)**.** *Prove the symmetric Gaussian Poincare inequality*

$$Var_\gamma(f) \leq \frac{1}{2}\mathbb{E}_\gamma|\nabla f|^2$$

for all even *locally-Lipschitz functions f on $\mathbb{R}^n$ by using the decomposition into Hermite polynomials (rather than by linearizing Blaschke-Santalo inequality like we did in class).*

Question 4.26 (1 point)**.** *Show that the Blaschke-Santalo inequality and Fathi's inequality are in fact equivalent (in class we only deduced the latter from the former).*

Question 4.27 (2 points)**.** *Prove the result of Saraglou.*
a) See the lecture notes for the definition of the log-addition. Show that the Log-Brunn-Minkowski inequality for Lebesgue measure

$$\left|\frac{K +_0 L}{2}\right| \geq \sqrt{|K| \cdot |L|}$$

(for any symmetric convex bodies K and L in $\mathbb{R}^n$) implies the Log-Brunn-Minkowski inequality for any even log-concave measure μ on $\mathbb{R}^n$ with full support:

$$\mu\left(\frac{K +_0 L}{2}\right) \geq \sqrt{\mu(K)\mu(L)}$$

(for any symmetric convex bodies K and L in $\mathbb{R}^n$). Conclude that the Log-Brunn-Minkowski conjecture implies the B-conjecture.
Hint: use the Prekopa-Leindler inequality.

b) Show the converse implication.
Hint: consider the situation near the origin and use the scale-invariance of the inequality in the Lebesgue case.

Question 4.28 (2 points)**.** *Confirm that the validity of the B-conjecture for all rotation-invariant log-concave measures is equivalent to the fact that for any even log-concave measure μ,*

$$\mu(RB_2^n)\mu\left(\frac{1}{R}B_2^n\right) \leq \mu(B_2^n)^2.$$

(recall that this corresponds to a very partial case and a sanity check in the Conjecture from Question 4.24.)

Question 4.29 (2 points)**.** *Show Klartag's theorem generalizing the functional Brunn-Minkowski inequality: for any even log-concave measure μ,*

$$\int e^{-\phi} d\mu \cdot \int e^{-\phi^*} d\mu \leq \left(\int e^{-\frac{x^2}{2}} d\mu\right)^2.$$

Hint: use Cafarelli's contraction theorem.

Question 4.30 (10 points)**.** *Attempt to make any progress on the "original B-conjecture": let $z \in \mathbb{R}^n$ and let K be a symmetric convex set in $\mathbb{R}^n$. Then the function*

$$\frac{\gamma(tK+z)}{\gamma(tK)}$$

is non-decreasing in $t \geq 1$. Here γ is the standard Gaussian measure.

Question 4.31 (2 points)**.** *Show that the B-theorem of Cordero-Erasquin, Fradelizi and Maurey would follow from the confirmation of the conjecture from Question 4.30.*

Hint: write the conclusion in terms of a non-negative derivative at $t = 1$; then note that the arising inequality implies that certain function which depends on $z \in \mathbb{R}^n$ is increasing along each ray, and therefore it is convex at the point $z = 0$. Consider the Laplacian in z.

Question 4.32 (2 points)**.** *Prove the result of Bobkov: the following are equivalent:*

- *For a symmetric convex body K of volume 1, the measure with the density*

$$\frac{1}{\sqrt{2\pi}^n \gamma(K)} e^{-\frac{x^2}{2}} 1_K(x) dx$$

  is isotropic.

- *For a symmetric convex body K of volume 1 and for any volume-preserving linear transformation T on $\mathbb{R}^n$, $\gamma(K) \geq \gamma(TK)$.*

Hint: use the B-theorem.

Question 4.33 (3 points)**.** *Prove the improved Log-Sobolev inequality: for any convex function V on $\mathbb{R}^n$ such that $\int e^{-V} = 1$,*

$$-\int Ve^{-V} \leq \frac{n}{2}\log\frac{\int \Delta V e^{-V}}{n} - n\log\sqrt{2\pi e}.$$

Question 4.34 (10 points)**.** *Is it possible to deduce from the Reverse Log-Sobolev inequality and/or the (generalized) Log-Sobolev inequality the following corollary of the Entropy Power Inequality?*

Let X and Y be any two centered random vectors in $\mathbb{R}^n$ and X' and Y' are independent centered Gaussians (whose covariance matrices are scalar), such that $h(X) = h(X')$ and $h(Y) = h(Y')$. Then

$$h(X+Y) \geq h(X'+Y'),$$

where

$$h(X) = -\int f\log f,$$

where f is the density according to which X is distributed.

Question 4.35 (2 points)**.** *Find Fathi's original proof for his inequality, which relies on the Reverse Log-Sobolev inequality (which we discussed) as well as the following fact (following from works of Cordero-Erasquin, Klartag and Santambrogio).*

Let μ be a centered probability measure whose support has non-empty interior. Then there exists an essentially continuous convex function φ, unique up to translations, such that $\rho = e^{-\varphi}dx$ is a probability measure on $\mathbb{R}^n$ whose push-forward by the map $\nabla\varphi$ is μ. Moreover, it satisfies

$$\rho = argmin\left\{-\frac{1}{2}W_2(\mu,\nu)^2 + Ent_\gamma(\nu)\right\}.$$

Clarification: do not aim to prove this fact, only aim for the implication of Fathi's theorem from this fact combined with the Reverse Log-Sobolev.

Question 4.36 (1 point)**.** *Suppose u, v on $\mathbb{R}^n$ are 2-homogeneous convex functions. Prove that*

$$\int e^{-\frac{u+v}{2}} det\left(\frac{\nabla^2 u + \nabla^2 v}{2}\right) \geq \sqrt{\int e^{-u} det(\nabla^2 u)\cdot\int e^{-v} det(\nabla^2 v)}.$$

Hint: use the fact that for a 2-homogeneous function, $2u = \langle \nabla u, x\rangle$ and the change of variables that we used when proving the Reverse Log-Sobolev inequality, together with the Prekopa-Leindler inequality.

Question 4.37 (1 point)**.** *Prove the conclusion of Question 4.24 under the assumption that both K and μ are unconditional.*

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