ICMU Mini-course

Limits of Discrete Structures

1. Introduction

Baby example: Graphs,

edge density $\rho(G) := \dfrac{e(G)}{\binom{v(G)}{2}} \in [0,1]$

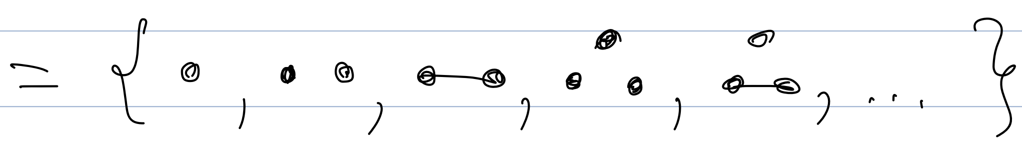
distance $\delta(H,G) := |\rho(H) - \rho(G)|$

completion $LIM \cong [0,1]$

$G \mapsto \rho(G) \in [0,1] \cap \mathbb{Q}$

2 Homomorphisms

$\mathcal{G} := \{$ finite graphs up to iso $\}$



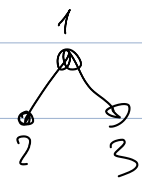
For graphs F, G, a map $f: V(F) \to V(G)$ is

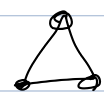
- **homomorphism** (hom) if $f(E(F)) \subseteq E(G)$
- **injective hom** if hom & injective
- **induced hom** if inj. hom &

$$f(E(\bar{F})) \subseteq E(\bar{G})$$

complement

Counts: $\hom(F,G)$, $\mathrm{inj}(F,G)$, $\mathrm{ind}(F,G)$

Ex P_3 

, K_3 

$$\hom(P_3, K_3) = 3 \cdot 2 \cdot 2$$

$$\mathrm{inj}(P_3, K_3) = 3 \cdot 2 \cdot 1$$

$$\mathrm{ind}(P_3, K_3) = 0$$

$$\mathrm{inj}(F,G) = \sum_{\substack{F' : V(F') = V(F) \\ E(F') \supseteq E(F)}} \mathrm{ind}(F', G)$$

$F' \supseteq F$

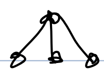
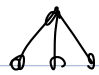
<u>Möbius inverse</u> of $f : \mathcal{G} \to \mathbb{R}$ is

$$f^{\uparrow}(F) := \sum_{F' \supseteq F} (-1)^{e(F') - e(F)} f(F')$$

<u>Lemma 2.1</u>: $\operatorname{ind}(F, -) = \operatorname{inj}^{\uparrow}(F, -)$.

<u>Ex</u> $\operatorname{ind}(\Lambda, -) = \operatorname{inj}(\Lambda, -) - \operatorname{inj}(\Delta, -)$

$$G \longmapsto \operatorname{ind}(F, G)$$

$\hom($$, G) = \operatorname{inj}($$, G) + 3 \operatorname{inj}(\Lambda, G)$
$+ \operatorname{inj}($$, G)$

Densities of k-vertex F in n-vertex G:

$$t(F, G) := \hom(F, G) / n^k$$

$$\left.\begin{aligned} t_{\mathrm{inj}}(F, G) &:= \operatorname{inj}(F, G) / (n)_k \\ t_{\mathrm{ind}}(F, G) &:= \operatorname{ind}(F, G) / (n)_k \end{aligned}\right\} n \geq k$$

$$(n)_k := n(n-1) \dots (n-k+1)$$

<u>Ex</u> $t(\Lambda, \Delta) = 3 \cdot 2 \cdot 2 / 3^3 = 4/9$

$t_{inj}(\Delta, \Delta) = 3\cdot 2\cdot 1 \;/ 3\cdot 2\cdot 1 = 1$

Ex: $$\mathbb{P}_{\substack{k\text{-set } X \\ \subseteq V(G)}} \left[G[X] \cong F \right] = \frac{k! \; t_{ind}(F, G)}{|\mathrm{aut}(F)|}$$

3 Covergence

$(G_n) = (G_1, G_2, \dots)$ s.t. $v(G_1) < v(G_2) < \dots$

Def (G_n) converges (to ϕ) if

$\forall F \in \mathcal{G}$ $(t(F, G_n))$ converges (to $\phi(F)$)

$[t_{inj}, t_{ind}]$

Ex
- $(K_n) \to \phi \equiv 1$
- (G_n) with $e(G_n) = o(v(G_n)^2)$

converges to $\phi(F) = \begin{cases} 0 & \text{if } e(F) \geq 1 \\ 1 & \text{o/w} \end{cases}$

Def (G_n) is p-quasirandom if

it converges to $\phi_p(F) := p^{e(F)}$

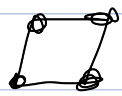
Lemma 3.1 $\forall p \in [0,1]$, $(G(n,p))$ is p-quasi-random with probability 1.

$G(n,p)$: $V = \{1, \dots, n\} =: [n]$

ij is edge w. probability p

<u>Thm 3.1</u> (Chung–Graham–Wilson '89)

$\forall (G_n)$ if $t($$, G_n) = p + o(1)$ &

$t($$, G_n) = p^4 + o(1)$ then

(G_n) is p-quasirandom.

<u>Limit version:</u> $\forall \phi \in LIM$

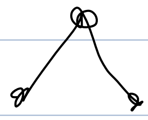
$\big(\phi($$) = p,\ \phi($$) = p^4 \Rightarrow \forall F\ \phi(F) = p^{e(F)} \big)$

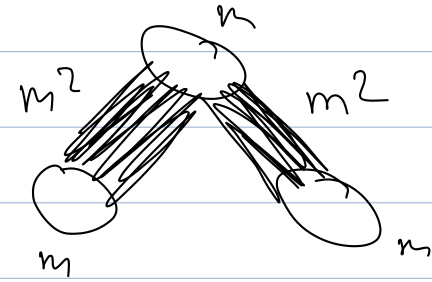
<u>Def</u> $LIM := \{ \phi : \mathcal{G} \to [0,1] : \exists (H_n) \to \phi \}$

$LIM \subseteq [0,1]^{\mathcal{G}}$, with product topology, closed, compact

<u>Lemma 3.2</u> $\forall (G_n)$ has a convergent subsequence. $\square$

$G \mapsto \lim_{m \to \infty} \phi_G$ of $(G(m))$ $\leftarrow$ m-blowup

P_2 

$P_2(m)$ 

$$t(F,G) = t(F, G(m))$$

Properties of $\phi \in LIM$:

- <u>normalised</u>: $\phi(K_1) = 1$
- multiplicative: $\phi(GH) = \phi(G)\phi(H)$

  disjoint union
- $\phi^\uparrow \geq 0$ $\left[\forall F \ \ \phi^\uparrow(F) = \lim\limits_{n\to\infty} \underbrace{t^\uparrow_{inj}(F,G_n)}_{\substack{t_{ind}(F,G_n) \\ \geq 0}}\right]$

<u>Thm 3.2</u> (Lovász-Szegedi '06)

$\phi : \mathcal{G} \to \mathbb{R}$ is in LIM if and only if normalised, multiplicative & $\phi^\uparrow \geq 0$.