Cluster Algebras

ICMU

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Course Overview

* Lecture 1: Defn and basic properties of cluster Algebras

* Lecture 2: Cluster algebras from surfaces

* Lecture 3: cluster algebras associated to Grassmannians $Gr(k,n)$

Overview

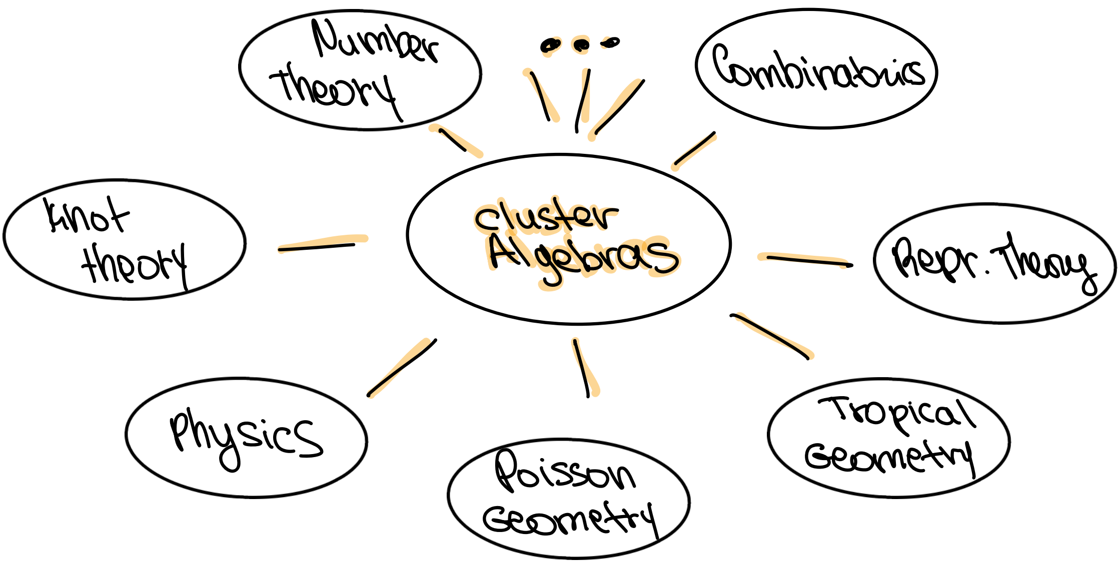
* Definition of cluster Algebras

* key properties

* classification

Cluster Algebras

* introduced by Fomin-Zelevinsky in 2001
* class of commutative rings in $\mathbb{Q}(x_1, \dots, x_n)$



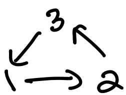
Cluster Algebras

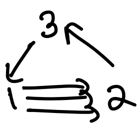
* a quiver Q is a (finite) directed graph without loops  and 2-cycles 

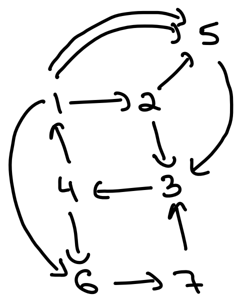
* label the vertices $1, 2, \ldots, n$

ex

$1 \to 2 \to 3$







Cluster Algebras

* quiver mutation is a local operation on a quiver defined as follows:

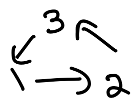
Given a vertex k of Q, let $\mu_k(Q)$ be obtained from Q by applying these steps:

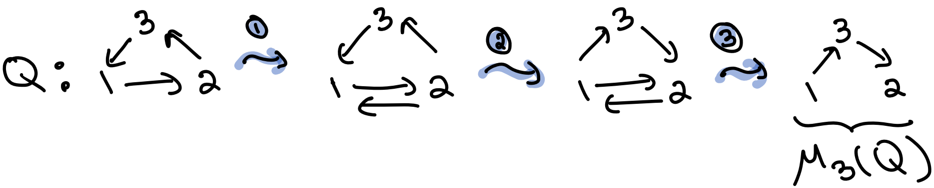
1) for every $i \to k \to j$ add $i \to j$
2) reverse all arrows at k
3) remove maximal collection of 2-cycles

$\mu_k(Q)$ is called mutation of Q at k

Cluster Algebras

* $\mu_k(Q)$ is obtained from Q as follows:
  1) for every $i \to k \to j$ add $i \to j$
  2) reverse all arrows at k
  3) remove maximal collection of 2-cycles

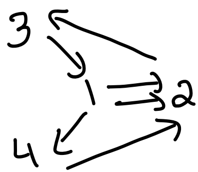
ex find $\mu_3(Q)$ where $Q =$ 

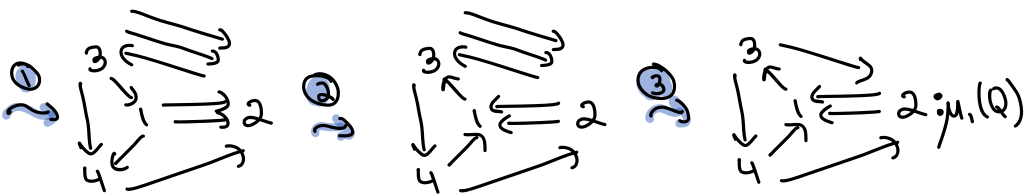


Cluster Algebras

* $\mu_k(Q)$ is obtained from Q as follows:
  1) for every $i \to k \to j$ add $i \to j$
  2) reverse all arrows at k
  3) remove maximal collection of 2-cycles

ex find $\mu_1(Q)$ Q:





Cluster Algebras

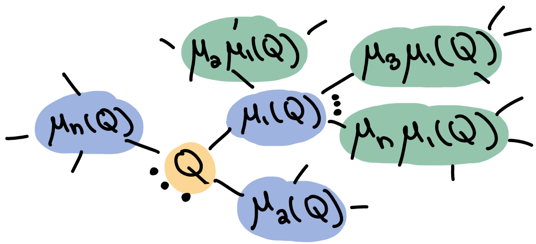
* Facts/observations:

1. Q and $\mu_k(Q)$ have same # of vertices
2. $\mu_k(Q)$ has no loops and 2-cycles
3. $\mu_k(Q)$ is a local operation on Q
4. $\mu_k(\mu_k(Q)) = Q$
5. $\mu_i(\mu_j(Q)) \neq \mu_j(\mu_i(Q))$ in general

Cluster Algebras

* Q is mutation equivalent to Q' if there exists a finite sequence of mutations μ_I s.t. $\mu_I(Q) = Q'$

* [Q] - set of all quivers mutation equivalent to Q



note: there is no known algorithm to determine whether two quivers are mutation equivalent

Cluster Algebras

* Q - quiver with vertices $1, \dots, n$

* **cluster algebra** $\mathcal{A}_Q \subset \mathbb{Q}(x_1, \dots, x_n)$ defined by a set of generators, called **cluster variables**, and satisfying **exchange relations** that are defined recursively

initial cluster variables

* **initial seed**: $(\mathbf{x} = (x_1, \dots, x_n), Q)$
* **mutation** μ_k of a seed $(\mathbf{x}, Q)$ is a new seed $\mu_k(\mathbf{x}, Q) = (\mu_k(\mathbf{x}), \mu_k(Q))$

* $\mathcal{A}_Q$ is generated by all cluster variables obtained from the initial seed by sequences of mutations

Cluster Algebras

Given a seed $(\mathbf{x}, Q)$ its mutation is defined as follows:

cluster variables

* let $\mathbf{x} = (f_1, \dots, f_n)$ where $f_i \in \mathbb{Q}(x_1, \dots, x_n)$

* $\mu_k(\mathbf{x}) = (f_1, \dots, f_{k-1}, f_k', f_{k+1}, \dots, f_n)$ where

$$f_k' = \frac{\prod\limits_{\substack{i \to k \\ \text{in } Q}} f_i + \prod\limits_{\substack{k \to j \\ \text{in } Q}} f_j}{f_k}$$

exchange relation

* $\mu_k(\mathbf{x}, Q) := (\mu_k(\mathbf{x}), \mu_k(Q))$

Cluster Algebras

ex $Q:\ 1 \to 2,\ 2 \to 3,\ 3 \to 1$ $\qquad \mathcal{A}_Q \subset \mathbb{Q}(x_1, x_2, x_3)$

initial seed: $\big((x_1, x_2, x_3),\ 1 \to 2 \to 3 \to 1\big)$

$\Big\downarrow \mu_1$

$\left(\left(\dfrac{x_2+x_3}{x_1}, x_2, x_3\right),\ 2 \to 1 \to 3\right)$ (the arrow between 2 and 3 is crossed out)

$\Big\downarrow \mu_2$

$\left(\left(\dfrac{x_2+x_3}{x_1}, \dfrac{\frac{x_2+x_3}{x_1}+1}{x_2}, x_3\right),\ 1 \to 2,\ 1 \to 3\right)$

$\mathcal{A}_Q$ is generated by $x_1, x_2, x_3, \dfrac{x_2+x_3}{x_1}, \dfrac{\frac{x_2+x_3}{x_1}+1}{x_2}, \ldots$

Cluster Algebras

* exchange graph of $\mathcal{A}_Q$ is n-regular graph s.t.

vertices ~ seeds under equivalence of permuting the order of cluster variables and applying an isomorphism on the quivers

edges ~ mutations

Cluster Algebras

ex $Q = \overset{1}{\bullet}$

$$((x_1), \bullet) \xrightarrow{\mu_1} \left(\left(\frac{1+1}{x_1}\right), \bullet\right)$$

$$\mathcal{A}_Q = \left\langle x_1, \frac{2}{x_1} \right\rangle = \mathbb{Q}[x_1^{\pm 1}]$$

ex $Q = \overset{1}{\bullet} \quad \overset{2}{\bullet}$

$$((x_1, x_2), \overset{1}{\bullet} \ \overset{2}{\bullet})$$

$$\xrightarrow{\mu_1} \left(\left(\frac{2}{x_1}, x_2\right), \overset{1}{\bullet} \ \overset{2}{\bullet}\right) \xrightarrow{\mu_2} \left(\left(\frac{2}{x_1}, \frac{2}{x_2}\right), \overset{1}{\bullet} \ \overset{2}{\bullet}\right)$$

$$\xrightarrow{\mu_2} \left(\left(x_1, \frac{2}{x_2}\right), \overset{1}{\bullet} \ \overset{2}{\bullet}\right) \xrightarrow{\mu_1} \left(\left(\frac{2}{x_1}, \frac{2}{x_2}\right), \overset{1}{\bullet} \ \overset{2}{\bullet}\right)$$

$$\mathcal{A}_Q = \left\langle x_1, x_2, \frac{2}{x_1}, \frac{2}{x_2} \right\rangle = \mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}]$$

Cluster Algebras

Laurent Phenomenon [Fomin-Zelevinsky 2001]

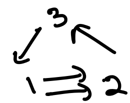
$$\mathcal{A}_Q \subset \mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$$

i.e. cluster variables are Laurent polynomials in x_i's

ex

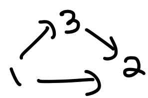
(x_1, x_2, x_3) $\overset{\mu_1}{\rightsquigarrow}$

$\left(\frac{x_2^2+x_3}{x_1}, x_2, x_3\right)$ $\overset{\mu_2}{\rightsquigarrow}$



$\left(\frac{x_2^2+x_3}{x_1}, \frac{(x_2^2+x_3)^2+x_3}{x_2}, x_3\right)$

$\Big\downarrow \mu_3$



$$\left(\frac{x_2^2+x_3}{x_1}, \frac{(x_2^2+x_3)^2+x_3}{x_2}, \frac{\frac{x_2^2+x_3}{x_1} + \frac{(x_2^2+x_3)^2+x_3}{x_2}}{x_3}\right)$$

$\overset{\mu_2}{\leftsquigarrow}$

$$\frac{\left(\frac{\frac{x_2^2+x_3}{x_1} + \frac{(x_2^2+x_3)^2+x_3}{x_2}}{x_3}\right)\cdot\left(\frac{x_2^2+x_3}{x_1}\right) + 1}{\frac{(x_2^2+x_3)^2+x_3}{x_2}}$$

Cluster Algebras

Positivity [Schiffler-Lee 2013]

$$\{\text{cluster variables}\} \subset \mathbb{Z}_{\geq 0}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$$

note:

$$\frac{\prod \text{positive Laurent polynomials} + \prod \text{positive Laurent polynomials}}{\text{positive Laurent polynomial}} \stackrel{?}{=} \text{positive Laurent polynomial}$$

ex

$$\frac{x^3+1}{x+1} = x^2 - x + 1$$

all positive

negative

Cluster Algebras

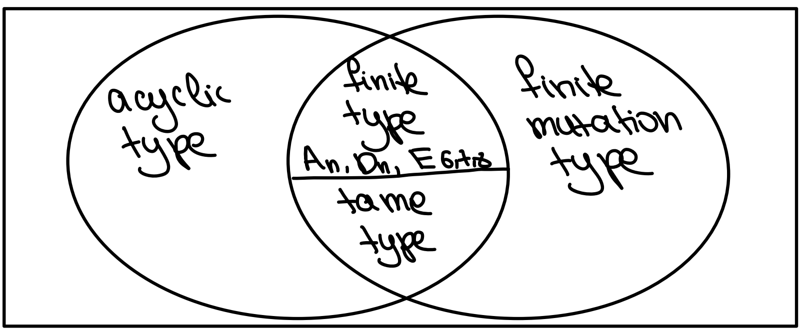
* Laurent phenomenon and positivity motivate the following questions:

Questions:

* can we find an explicit formula for cluster variables?
* what are the coefficients in a cluster variable counting?
* can we find a "nice" basis of a cluster algebra? what do structure constants represent?

Cluster Algebras

* $\mathcal{A}_Q$ is of finite type if there are finitely many cluster variables
* $\mathcal{A}_Q$ is of finite mutation type if $[Q]$ is finite
* $\mathcal{A}_Q$ is acyclic if there exists Q' in $[Q]$ that has no oriented cycles



Cluster Algebras

* **Thm** [Fomin-Zelevinsky]

$\mathcal{A}_Q$ is of finite type iff Q is mutation equivalent to one of the following

A_n : $1 \to 2 \to 3 \to \cdots \to n$

D_n : $1 \to 2 \to 3 \to \cdots \to n-2$, with $n-2 \to n-1$ and $n-2 \to n$

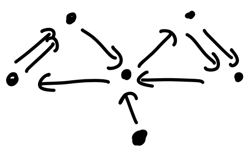
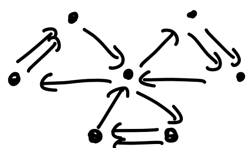
$E_{6,7,8}$: $1 \to 2 \to 3 \to 4 \to 5 \to 7 \to 8$, with $3 \downarrow 6$

Cluster Algebras

* **Thm** [Felikson-Shapiro-Tumarkin]

A Q is of finite mutation type iff Q is one of the following:

- $1 \overset{\vdots}{\rightrightarrows} 2$
- comes from triangulation of a surface
- $E_{6,7,8}$, $\tilde{E}_{6,7,8}$, $E^{(1,1)}_{6,7,8}$
- X_6, X_7

ex $X_6 =$  $X_7 =$ 

Cluster Algebras

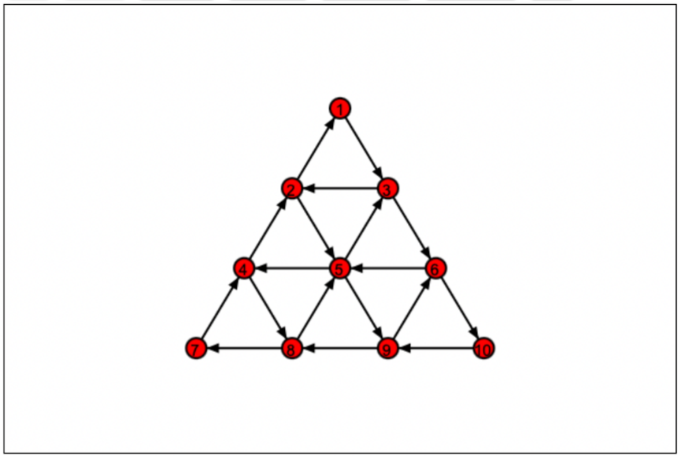
* Keller's mutation applet

Quiver mutation in JavaScript

New quiver ... | Valued arrows | Live quiver | Renumber | History | Random ... | Repeat random

Add framing | Hide frozen | Center

Back | Forward | Add nodes | Add arrows | Delete nodes | Freeze nodes | Done



Cluster Algebras

* there are various generalizations of cluster algebras

  - skew-symmetrizable case
  - coefficients
  - quantum cluster algebras
  - generalized exchange relations
  - "super" cluster algebras

  ⋮

Cluster Algebras

* by specializing cluster variables $x_i = 1$, cluster variables evaluate to positive integers, so what sequences can we get in this way?
  Markov numbers, Somos sequences, Fibonacci sequence, ...
* cluster algebras can be used to compute knot invariants
* cluster algebras were used to show Zamolodchikov periodicity conjecture
* cluster algebras from $Gr(k,n)$ can be used to understand asymptotics of solutions to KP equation
* ....

Thank you!