

ICMU Bayes Intro, Homework 3
Due on October 3rd, 2025

1. Prove that $\text{Binomial}(N, \frac{1}{2})$ is a stationary distribution of the Ehrenfest diffusion Markov chain.
Hint: use the formula for the Binomial probability mass function and check detailed balance.
2. Prove that the Rosenbluth-Hastings algorithm results in a Markov chain with the desired distribution (target distribution) being its stationary distribution.
Hint: check detailed balance; since R-H acceptance ratio can be either ≤ 1.0 or > 1.0 , it is easier to consider these cases separately.
3. Show that the uniform random walk proposal density, which we used in MCMC for the standard normal distribution in class, is symmetric: $q(x | y) = q(y | x)$. Recall that given the current state of MCMC x this proposal generates $Y = x + U$, where $U \sim \text{Unif}(-\delta, \delta)$.
4. Let's return to the Poisson-Gamma model for text/sms messages that we have looked at in homework assignments 1 and 2:

$$\lambda \sim \text{Gamma}(400, 80),$$
$$Y_1, \dots, Y_n \stackrel{\text{iid}}{\sim} \text{Poisson}(\lambda).$$

We observed $(y_1, \dots, y_n) = (0, 1, 1, 1, 3, 3, 2, 6, 5, 2)$ and derived the following result for the posterior distribution of λ :

$$p(\lambda | y_1, \dots, y_n) \propto \lambda^{s-1+\sum_{i=1}^n y_i} e^{-(n+r)\lambda}. \quad (1)$$

Suppose we didn't recognize that the above posterior corresponds to the Gamma density. So instead of the analytical solution, we will use MCMC with the uniform random walk proposal (same as in problem 3) to approximate the posterior (1).

- (a) What is the Rosenbluth-Hastings acceptance ratio in this case?
- (b) Implement a Rosenbluth-Hastings algorithm with a uniform random walk proposal and compute posterior mean and 95% Bayesian credible intervals for the Poisson rate λ .
- (c) Experiment with different numbers of MCMC iterations and values of the tuning parameter δ of the uniform random walk proposal and comment on the agreement/disagreement of your MCMC results and analytic-based results that you had obtained in homework assignment 2.
- (d) In addition to the posterior summaries above, provide a trace plot and a histogram of the MCMC samples.