Cluster Algebras

ICMU

June 24-28, 2024

Khrystyna Serhiyenko

Course Overview

* Lecture 1: Defn and basic properties of cluster Algebras
* Lecture 2: Cluster algebras from surfaces
* Lecture 3: cluster algebras associated to Grassmannians $Gr(k,n)$

Overview

* Surface triangulations
* cluster algebras from surfaces
* expansion formula

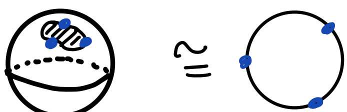
Marked Surfaces

* S - compact connected oriented 2-dim Riemann surface (up to homotopy)

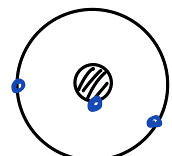
sphere genus 0 — torus genus 1 — genus 2

* (S, M) - surface S with boundary components and a set M of marked points with at least one marked point on each boundary component

torus with 2 boundary components



sphere with 1 boundary component $\cong$ disk

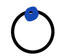
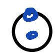
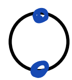
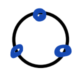


annulus

Marked Surfaces

* marked points in the interior of S are called punctures

* for technical reasons we require that (S, M) is not one of the following:

  - sphere with ≤ 3 punctures 
  - monogon with 0 or 1 punctures  
  - digon w/o punctures 
  - triangle w/o punctures 

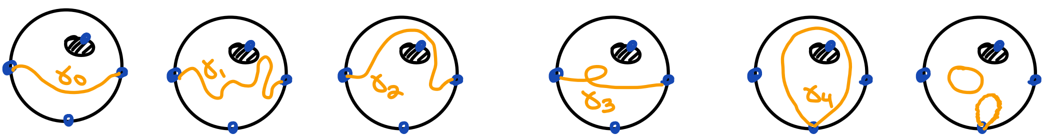
Marked Surfaces

* $\boxed{\text{arc}}$ γ in (S, M) is a curve in S such that
  - endpoints of γ are in M
  - except for endpoints γ is disjoint from M and ∂S
  - γ does not cut out an unpunctured monogon or bigon

    i.e. γ not contractable into M or a single boundary segment of ∂S
  - γ does not intersect itself

* consider arcs up to isotopy (fixing endpoints)

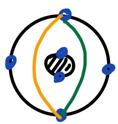
ex



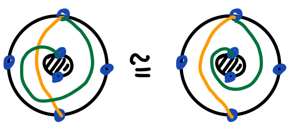
arc | arc | not an arc | not an arc | arc | not arcs

Marked Surfaces

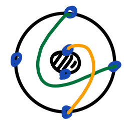
* γ and γ' are compatible if they do not intersect in the interior of S
  i.e. there are curves $\alpha \sim \gamma$ and $\alpha' \sim \gamma'$ s.t. α and α' do not cross except their endpoints may coincide



compatible

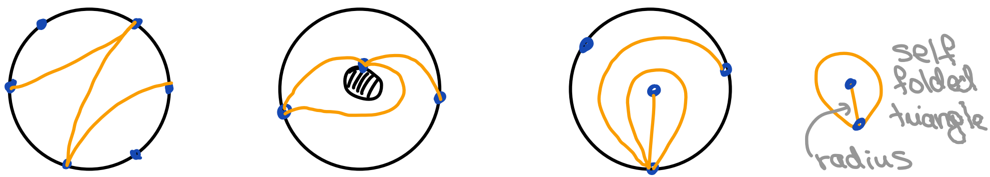


compatible



not compatible

* triangulation T of (S,M) is a max. collection of pairwise compatible arcs. Arcs of T subdivide the surface into triangles

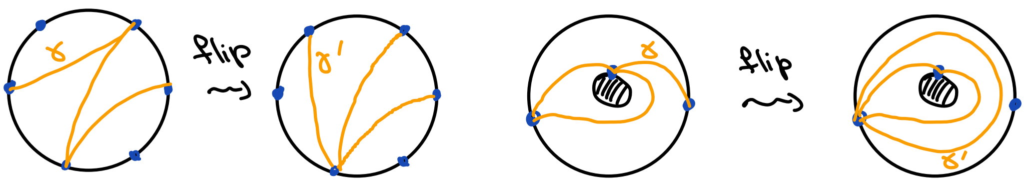


Marked Surfaces

* Let $\gamma \in T$ s.t. γ is not a radius of a self-folded triangle, a **flip** of T at γ is a unique new triangulation $T' = T \setminus \{\gamma\} \cup \{\gamma'\}$

note:  cannot be flipped

ex

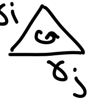


Th: any two triangulations of (S, M) are connected via a sequence of flips

Cluster Algebras

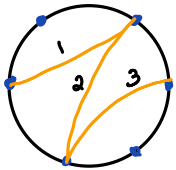
* (S,M) is w/o punctures (for simplicity)
* T triangulation of (S,M) $\rightsquigarrow$ quiver Q_T

  arcs γ_i of T — vertices i

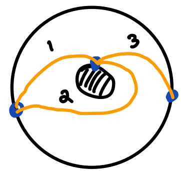


$i \to j$

ex



$\rightsquigarrow$ $1 \leftarrow 2 \rightarrow 3$



$\rightsquigarrow$ $1 \to 2$, $3 \to 2$, $1 \to 3$

Prop flip of T at γ_i corresponds to quiver mutation of Q_T at i

$$\begin{tikzcd} T \arrow[r, squiggly] \arrow[d, "\text{flip at } \gamma_i"'] \arrow[dr, phantom, "\circlearrowleft"] & Q_T \arrow[d, "\text{mutate at } i"] \\ T' \arrow[r, squiggly] & Q_{T'} \end{tikzcd}$$

Cluster Algebras

* (S, M) surface with triangulation T, then the surface cluster algebra associated to T is $\mathcal{A} = \mathcal{A}_{Q_T}$

Th there are bijections

$$\left\{ \begin{array}{l} \text{cluster variables} \\ x_\gamma \text{ of } \mathcal{A} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{arcs } \gamma \text{ of} \\ (S, M) \end{array} \right\}$$

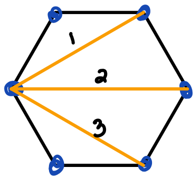
$$\left\{ \begin{array}{c} \text{seeds of } \mathcal{A} \\ ((x_{\gamma_1}, .., x_{\gamma_n}), Q_T) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{triangulations of } (S,M) \\ T = \{\gamma_1, .., \gamma_n\} \end{array} \right\}$$

$$\text{mutations} \longleftrightarrow \text{flips}$$

note $\mathcal{A}$ has infinitely many cluster variables, but it is of finite mutation type

Cluster Algebras

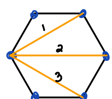
ex (S, M) - disk with 6 marked points on the boundary
i.e. hexagon



$\rightsquigarrow Q_T : 1 \to 2 \to 3$ type A_3 quiver

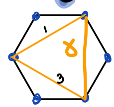
diagonals in hexagon $\leftrightsquigarrow$ cluster variables $\mathcal{A}_{1\to 2\to 3}$ $\Rightarrow \mathcal{A}_{1\to2\to3}$ is of finite type

triangulations of hexagon $\leftrightsquigarrow$ seeds of $\mathcal{A}_{1\to2\to3}$



$\leftrightsquigarrow$ $\{x_1, x_2, x_3\}$

$\downarrow$ flip $\qquad\qquad$ $\downarrow$ mutation

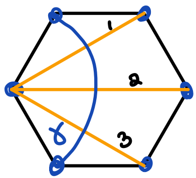


$\leftrightsquigarrow$ $\{x_1, x_\gamma = \dfrac{x_1 + x_3}{x_2}, x_3\}$

Expansion Formula

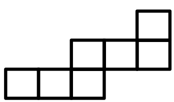
* explicit formula for each cluster variable x_γ

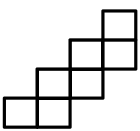
i.e.



obtain an expression for x_γ directly from the surface w/o having to perform flips/mutations

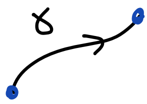
* snake graph $\mathcal{G}$ is a connected graph consisting of tiles □, where each tile is glued to the north or east edge of the previous tile





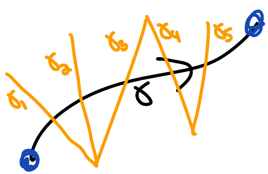
Expansion Formula

* fix triangulation T of a surface (S, M) and an arc $\gamma \notin T$

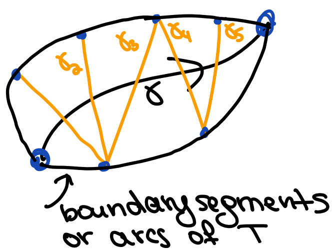


choose orientation of γ and a representative of γ that has a minimal # of crossings with arcs in T

* then γ crosses arcs of T $\gamma_1, \gamma_2, \ldots, \gamma_d$ in that order (where γ_i's do not have to be distinct)



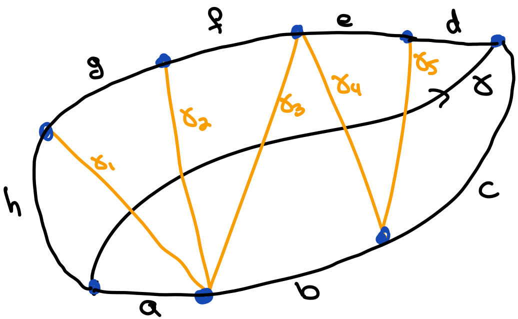
$\Rightarrow$

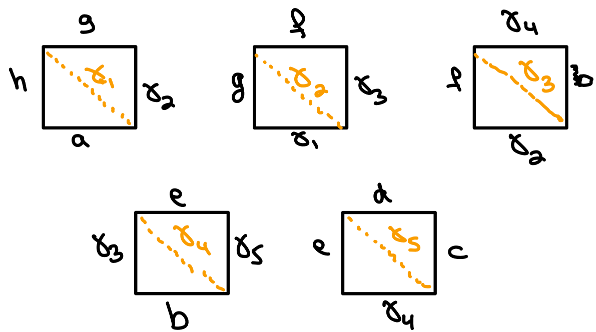


portion of the surface with the triangulation

Expansion Formula

* for each γ_i obtain a tile  that corresponds to the quadrilateral in the surface that contains γ_i as a diagonal

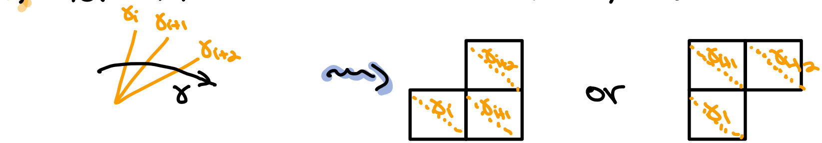


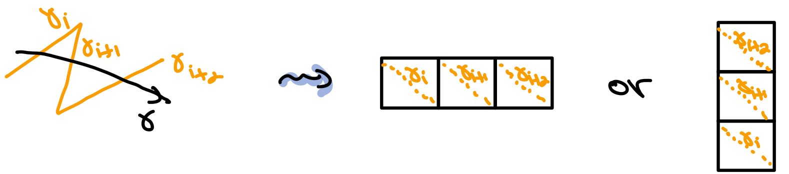


* now want to glue the tiles to obtain a snake graph

Expansion Formula

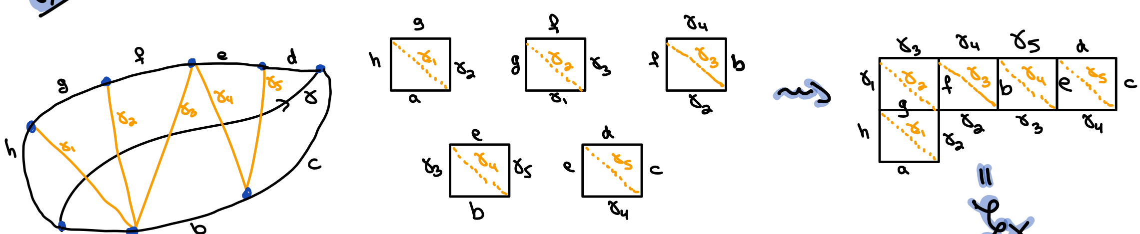
* for three consecutive $\gamma_i, \gamma_{i+1}, \gamma_{i+2}$ do the following





⊛ S.T. may need to flip the orientation of the tiles

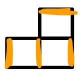
ex



$\mathcal{G}_\gamma$

Expansion Formula

* A **perfect matching** of a graph $\mathcal{G}$ is a subset P of the edges of $\mathcal{G}$ s.t. each vertex of $\mathcal{G}$ is incident to exactly one edge of P

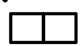
P 

 no perfect matching

note not every graph admits a perfect matching, but snake graphs always admit multiple

ex

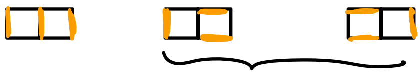
  

all perfect matchings of 

Expansion Formula

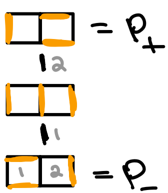
Q: how to find all perfect matchings of a snake graph $\mathcal{G}$?

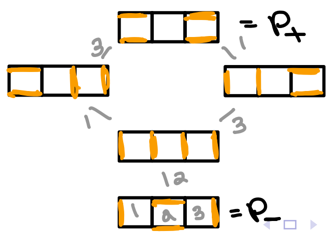
* $\mathcal{G}$ has exactly two perfect matchings P_+ and P_- that contain only boundary edges

ex 

P_+ and P_-

* start with minimal matching P_- and construct a poset by performing interchanges  when possible until reaching P_+, this gives a ~~poset~~

ex 



Expansion Formula

* T is a triangulation of (S, M) and γ is an arc not in T
* $T = \{\gamma_1, \ldots, \gamma_n\}$
* $\mathcal{G}_\gamma$ - associated snake graph
* $\mathrm{Match}(\mathcal{G}_\gamma)$ – set of perfect matchings of $\mathcal{G}_\gamma$
* $P \in \mathrm{Match}(\mathcal{G}_\gamma)$, define $x(P) = \prod x_i$ where the product is over all edges with label γ_i in P with multiplicities
* $\mathrm{cross}(\gamma) = \prod x_i$ where the product is over all γ_i's that γ crosses with multiplicities

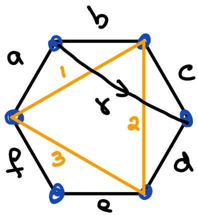
Expansion Formula [Musiker - Schiffler - Williams]

$$\boxed{x_\gamma = \frac{1}{\mathrm{cross}(\gamma)} \sum_{P \in \mathrm{Match}(\mathcal{G}_\gamma)} x(P)}$$

Corollary: positivity for cluster algebras from surfaces

Expansion Formula

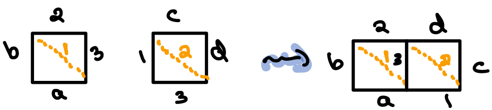
ex



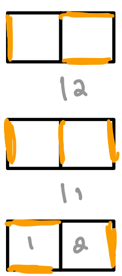
Find x_γ?

1. find $\mathcal{G}_{T_\gamma}$
2. find Match($\mathcal{G}_{T_\gamma}$)
3. apply Expansion Formula

① $\mathcal{G}_{T_\gamma}$ will have two tiles



② Match($\mathcal{G}_{T_\gamma}$)



x(P)

x_1

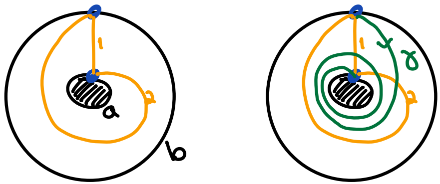
x_3

x_2

③ $x_\gamma = \frac{1}{x_1 x_2}(x_1 + x_2 + x_3)$

Expansion Formula

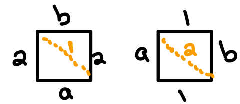
ex



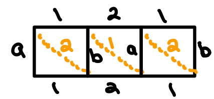
(S, M) and T

Find x_γ?

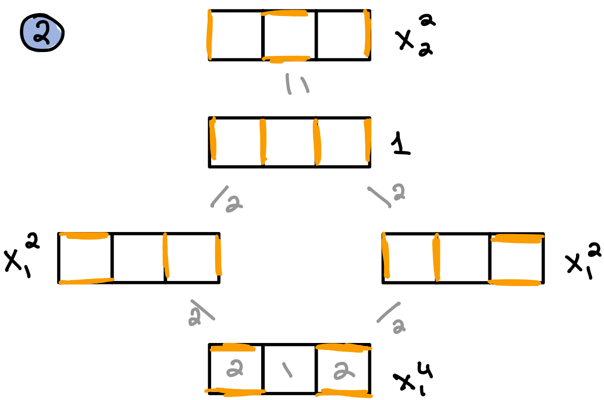
① tiles in $\mathcal{G}_\gamma$ are 2, 1, 2



$\mathcal{G}_\gamma =$



②

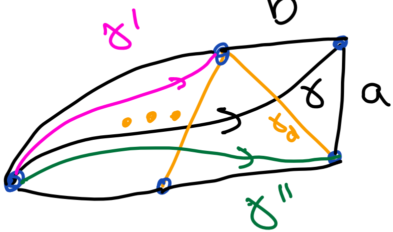


③

$$x_\gamma = \frac{1}{x_1 x_2^2}\left(x_1^4 + 2x_1^2 + x_2^2 + 1\right)$$

Expansion Formula

* proof idea for the expansion formula:
  - ~> induct on the # of crossings of γ and arcs in T



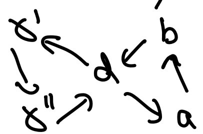
① γ crosses $\gamma_d \in T$ last

② γ', γ'' have fewer crossings with arcs of T then γ, so induction hypothesis holds

③ let T' be a triangulation containing $\gamma', \gamma'', \gamma_d, a, b$

can flip γ_d in T' to obtain γ to obtain x_γ in $\mu_d(x_{T'}, Q_{T'})$

in $Q_{T'}$



④ $$x_\gamma = \frac{x_b \cdot x_{\gamma''} + x_a x_{\gamma'}}{x_d}$$

$$x_\gamma x_d = x_b x_{\gamma''} + x_a x_{\gamma'}$$

⑤ to prove the expansion formula it suffices to check ④ holds in terms of matchings.

Bases

* elements of $\mathcal{A}_Q$ are polynomials in cluster variables, can we find a basis for $\mathcal{A}_Q$ over $\mathbb{Q}$.

* in particular want to write every element of $\mathcal{A}_Q$ uniquely

$$x_k x_{k'} = \prod_{i \to k} x_i + \prod_{k \to i} x_i$$ exchange relation

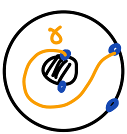
i.e. want to throw out relations

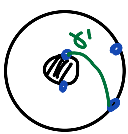
* $\mathcal{A}_Q$ has sums and products of cluster variables

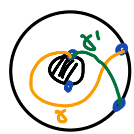
$x_\gamma \cdot x_{\gamma'}$ and $x_\gamma + x_{\gamma'}$

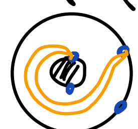
so we want to understand these operations geometrically

* sums can view as formal addition and products as superposition

$\gamma + \gamma' =$ 

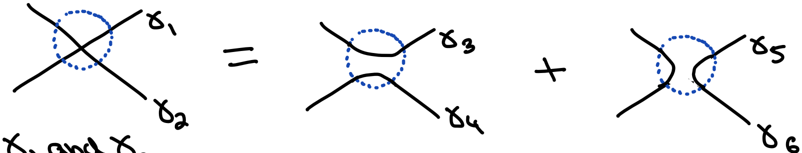
$+$ 

$\gamma \cdot \gamma' =$ 

$\gamma^2 =$ 

Bases

* the relations b/w cluster variables may be interpreted on level of arcs as **smoothing operation**:



γ_1 and γ_2 cross

Th $x_{\gamma_1} \cdot x_{\gamma_2} = x_{\gamma_3} x_{\gamma_4} + x_{\gamma_5} x_{\gamma_6}$

note via smoothing can get self-crossing arcs and closed loops, so need to set the following

if γ is contractible closed loop

set $x_\gamma = -2$

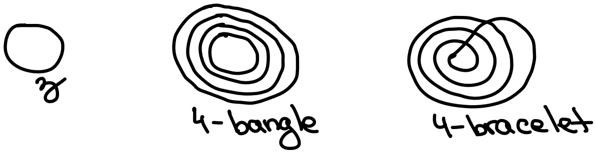
if γ has a contractible kink

set $x_\gamma = -x_{\bar{\gamma}}$

Bases

* a loop is **essential** if it is not contractiblo and has no self crossings

* for essential loop z define **k-bangle** and **k-bracelet** as follows



Def * B^o be the set of all products $\prod_{\gamma \in C} x_\gamma$ where C ranges over all multicurves of arcs and essential loops w/o crossings (contains bangles but not bracelets)

* B be the set of all products $\prod_{\gamma \in C} x_\gamma$ where C ranges over all collections of arcs and bracelets s.t. no crossings b/w elements of C except for self-crossings in bracelets and at most one bracelet for a given essential loop in C

Th [Musiker-Schiffler-Williams] B and B^o are bases of $\mathcal{A}$.

Thank you !