Lecture 3

I Legendre transform

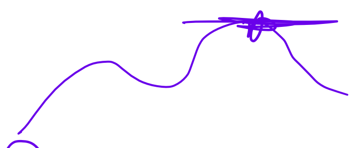
$$f^*(x) = \sup_{y \in \mathbb{R}^n} \left(\langle x, y \rangle - f(y) \right)$$

Examples

1) $\boxed{f(t) = \frac{t^p}{p}}$ $\quad p > 1 \quad$ on $\mathbb{R}$

$$f^*(t) = \sup_{s \in \mathbb{R}} \left(ts - f(s) \right) = \sup_{s \in \mathbb{R}} \left(ts - \frac{s^p}{p} \right) =$$

How to find maximum of a function?



Derivative!

$$\left(ts - \frac{s^p}{p} \right)'_s = t - s^{p-1} = 0$$

$$s_0 = t^{\frac{1}{p-1}}$$

$ts - \frac{s^p}{p}$

HW

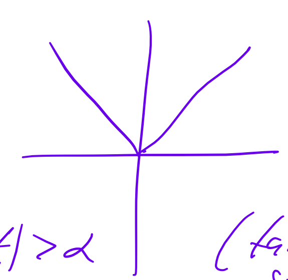
$$= t \cdot t^{\frac{1}{p-1}} - \frac{t^{\frac{p}{p-1}}}{p} = t^{\frac{p}{p-1}} \cdot \left(1 - \tfrac{1}{p}\right)$$

$$\boxed{f^*(t) = \frac{t^q}{q}} \quad \text{where} \quad q = \frac{p}{p-1}$$

$$\frac{1}{p} + \frac{1}{q} = 1$$

$$p, q > 1$$

2) $f(t) = \alpha |t| \qquad \alpha > 0$



$|t| > \alpha \qquad$ (take

∞

$$f^{*}(t) = \sup_{s\in\mathbb{R}} \left(st - \alpha |s| \right) = \begin{cases} \dots & (s=\infty) \\ 0 & |t| \le \alpha \quad (\text{take } s=0) \end{cases}$$

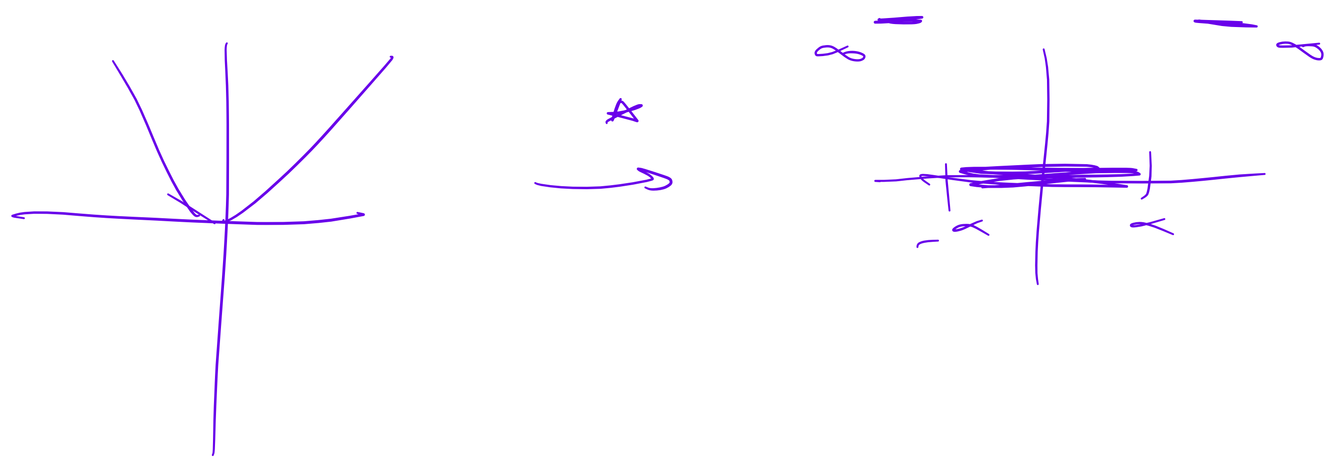
$$\| $$

$$\sup_{s\in\mathbb{R}} \left(|s| \cdot \left(\underline{\underline{\operatorname{sgn}(s)}} \cdot t - \alpha \right) \right)$$

$$\|$$

$$\sup_{s\in\mathbb{R}} \left(\underline{\underline{|s|}} \cdot \left(\underbrace{|t| - \alpha}_{} \right) \right)$$

Conclusion $\quad (\alpha \cdot |t|)^{*} = \begin{cases} \infty & \text{IF } |t| > \alpha \\ 0 & \text{IF } |t| \le \alpha \end{cases} = \mathbb{1}^{\infty}_{[-\alpha,\alpha]}$



(3) More generally $\quad K \subset \mathbb{R}^n$ convex body

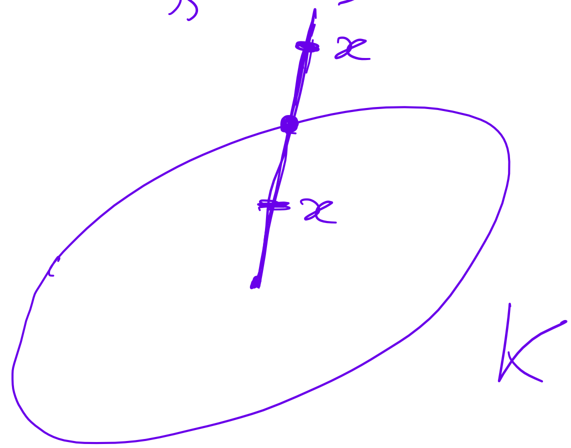
$$h_K(x) = \sup_{y\in K} \langle x, y \rangle$$

$$h_K^{*} = \mathbb{1}^{\infty}_K \,, \qquad \mathbb{1}_K^{*} = h_K$$

(4) Minkowski functional of K

$$\|x\|_K := \inf\{s > 0 : \tfrac{x}{s} \in K\}$$

$$\left(\frac{\|x\|_K^p}{p}\right)^* = \frac{\|x\|_{K^\circ}^q}{q}$$

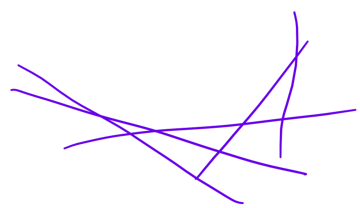


$p > 1$

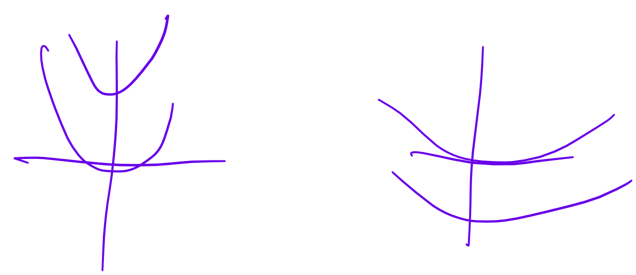
$$K^\circ = \{y \in \mathbb{R}^n : \forall x \in K,\ \langle x, y\rangle \le 1\}$$

Key properties

- $f : \mathbb{R}^n \to \mathbb{R}$ we have f^* is convex
- If f is convex $\Longrightarrow$ $f^{**} = f$



- $(f + a)^*(x) = f^* - a$
- $(af)^*(x) = a f^*\left(\frac{x}{a}\right)$, $a > 0$



Legendre transform of smooth functions

$$\boxed{f^*(x) = \sup_{y \in \mathbb{R}^n} \left(\langle x, y\rangle - f(y)\right)}$$

(HW)

Find sup by taking derivatives, $y_0 = \nabla f(x)$

$$\boxed{f(x) + f^*(\nabla f(x)) = \langle x, \nabla f(x)\rangle}$$

$\|$

$$\boxed{\nabla f \circ \nabla f^* = Id \quad \text{i.e.} \quad \nabla f(\nabla f^*(x)) = x}$$

$$\Downarrow$$

$$\boxed{\nabla^2 f^*(\nabla f(x)) = \left(\nabla^2 f(x)\right)^{-1}}$$

II Generalized Log-Sobolev inequality

Prekopá-Leindler inequality, f, g convex

$$\int e^{-((1-t)f+tg)^*} \geq \left(\int e^{-f^*}\right)^{1-t} \left(\int e^{-g^*}\right)^t$$

$$\log \int e^{-((1-t)f+tg)^*} \geq (1-t) \log \int e^{-f^*} + t \log \int e^{-g^*}$$

∇
0

$\log \int e^{-f^*}$ is concave

$$\boxed{\alpha(t) = \log \int e^{-((1-t)f+tg)^*} - (1-t) \log \int e^{-f^*} - t \log \int e^{-g^*}}$$

$\alpha(t) \geq 0$ on $t \in [0,1]$

$\alpha(0) = 0$

$\alpha'(0) \geq 0$

$$\boxed{\frac{d}{dt}\log\int e^{-((1-t)f+tg)^*}+\log\int e^{-f^*}-\log\int e^{-g^*}\geq 0}$$

$$\left(\log\varphi(t)\right)'=\frac{\varphi'(t)}{\varphi(t)}$$

$$\frac{d}{dt}\log\int e^{-((1-t)f+tg)^*}\Big|_{t=0}=\frac{\frac{d}{dt}\int e^{-((1-t)f+tg)^*}}{\int e^{-((1-t)f+tg)^*}}\Bigg|_{t=0}$$

$$=\boxed{\frac{-\int e^{-f^*}\cdot\frac{d}{dt}\left[((1-t)f+tg)^*\right]}{\int e^{-f^*}}}\quad \ominus$$

How to differentiate under Legendre transform?

Key Lemma

Let $V_t(x)$ be a family of functions $x\in\mathbb{R}^n$ indexed by t for $t\in[0,1]$ s.t. $V_t(x)\in C^2(\mathbb{R}^n,(0,1))$ and suppose that $V_t(x)$ is convex. Then

① $$\frac{d}{dt}V_t^*(x)=-\dot V_t(\nabla V_t^*(x))$$

HW ② $$\frac{d^2}{dt^2}V_t^*(x)=-\ddot V_t(\nabla V_t^*(x))+\left\langle(\nabla^2V_t(x))^{-1}\nabla\dot V_t\big|_{\nabla V_t^*},\nabla\dot V_t\big|_{\nabla V_t^*}\right\rangle$$

Hint for ① use $V_t(x)+V_t^*(\nabla V_t(x))=\langle x,\nabla V_t(x)\rangle$

differentiate in t.

... Going back to our computation

$$\frac{d}{dt}\left[\left((1-t)f+tg\right)^*\right] = -\dot{V}_t(\nabla V_t^*) \ominus$$

$$\ominus\ f(\nabla f^*) - g(\nabla f^*)$$

$$V_t = (1-t)f + tg$$

$$\dot{V}_t = g-f$$

$$\nabla V_t^*\Big|_{t=0} = \nabla f^*$$

$$\ominus\ \frac{-\int e^{-f^*}\left(f(\nabla f^*)-g(\nabla f^*)\right)}{\int e^{-f^*}}$$

$F = f^*$, $f = F^*$

$G = g^*$, $g = G^*$

$$\boxed{\frac{\int\left(G^*(\nabla F)-F^*(\nabla F)\right)e^{-F}}{\int e^{-F}} + \log\int e^{-F} - \log\int e^{-G} \geq 0}$$

Theorem (Minkowski's first inequality

for functions

F, G convex functions s.t. $\int e^{-F} = \int e^{-G}$.

Then
$$\int G^*(\nabla F)e^{-F} \geq \int F^*(\nabla F)e^{-F}$$

(HW) Consider $F(x) = \mathbb{1}_K^\infty$, $G(x) = \mathbb{1}_L^\infty$, get

$$V_1(K,L) \geq |K|^{\frac{n-1}{n}} \cdot |L|^{1/n}$$

Theorem (Generalized Log-Sobolev inequality)

F, G convex functions

$$\int G^*(\nabla F)e^{-F} \geq n\int e^{-F} - \int F e^{-F} - \int e^{-F} \cdot \log \frac{\int e^{-F}}{\int e^{-G}}$$

Proof

We know $\int G^*(\nabla F)e^{-F} - \int F^*(\nabla F)e^{-F} + \int e^{-F} \cdot \log \frac{\int e^{-F}}{\int e^{-G}} \geq 0$

We need to verify that

$$n\int e^{-F} - \int F e^{-F} = \int F^*(\nabla F)e^{-F}$$

Recall

$$F + F^*(\nabla F) = \langle \nabla F, x \rangle$$

$$F^*(\nabla F) = -F + \langle \nabla F, x \rangle$$

All we need: $\int \langle \nabla F, x \rangle e^{-F} = n\int e^{-F}$

$$-\int \langle \nabla e^{-F}, x \rangle = n\int e^{-F}$$

Claim $\boxed{n\int\varphi = -\int\langle\nabla\varphi, x\rangle}$ integration by parts ☒

Reformulation: $\varphi = e^{-F}$

$$\int G^*\left(-\frac{\nabla\varphi}{\varphi}\right)\cdot\varphi \geq n\int\varphi + \mathrm{Ent}(\varphi)$$

$$\mathrm{Ent}(\varphi) = \int\varphi\log\varphi - \int\varphi\cdot\log\int\varphi$$

$$\Downarrow \quad \boxed{G(x) = \frac{|x|^2}{2} = G^*(x)}$$

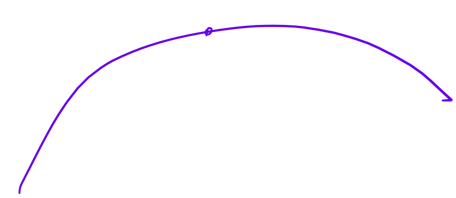
Thm (Gaussian Log-Sobolev inequality)

$$\mathrm{Ent}_\gamma(g^2) \leq 2\int|\nabla g|^2\,d\gamma$$

$$d\gamma = \frac{1}{\sqrt{2\pi}^{\,n}}e^{-\frac{|x|^2}{2}}, \quad \mathrm{Ent}_\gamma(F) = \int F\log F\,d\gamma - \int F\,d\gamma\cdot\log\int F\,d\gamma$$

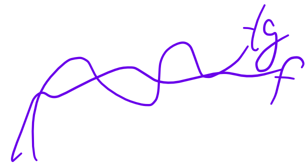
III The Brascamp-Lieb inequality

PL $\log\int e^{-f^*}$ is concave



$$\alpha''(0) \leq 0$$

$$f_t = f + tg$$



$$\boxed{\frac{d^2}{dt^2}\log\int e^{-f_t^*}\le 0}$$

(11)

$$\frac{d}{dt}\left[\frac{-\int \frac{d}{dt}f_t^*\cdot e^{-f_t^*}}{\int e^{-f_t^*}}\right]=\frac{\int e^{-f_t^*}\cdot\left(-\int\frac{d^2}{dt^2}f_t^* e^{-f_t^*}+\int\left(\frac{d}{dt}f_t^*\right)^2 e^{-f_t^*}\right)}{\left(\int e^{-f_t^*}\right)^2}$$

$$-\frac{\left(\int\left(\frac{d}{dt}f_t^*\right)^2 e^{-f_t^*}\right)^2}{\left(\int e^{-f_t^*}\right)^2}\Bigg|_{t=0}\le 0$$

$$\frac{d}{dt}f_t^*\Big|_{t=0}=-g(\nabla f)$$

$$\frac{d^2}{dt^2}f_t^*\Big|_{t=0}=\langle(\nabla^2 f(\nabla f))^{-1}\nabla g(\nabla f),\nabla g(\nabla f)\rangle$$

Change notation $\varphi=g(\nabla f)$

$f^*=V$

(hw)

Theorem (Brascamp-Lieb inequality)

$$\int\varphi^2 e^{-V}-\left(\int\varphi e^{-V}\right)^2\le\int\langle(\nabla^2 V)^{-1}\nabla\varphi,\nabla\varphi\rangle e^{-V}$$

$\int e^{-V} = 1$, V convex

$e^{-V}dx = d\mu$ ← log-concave measure

$$\mu(\lambda K + (1-\lambda)L) \geq \mu(K)^{\lambda}\mu(L)^{1-\lambda}$$

$K \approx \cdot\, x$

$L \approx \cdot\, y$