# Finding roots of complex polynomials: from numerical analysis to dynamical systems and computer algebra

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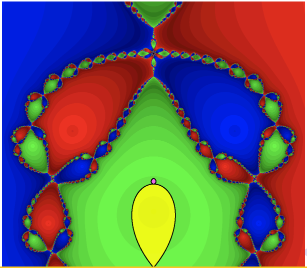


## 2 The fundamental theorem for algebra

**Theorem 1 (Gauss)**

*Every (univariate) degree $d$ polynomial over $\mathbb{C}$ has exactly $d$ roots, counting multiplicity.*

**Theorem 2 (Ruffini–Abel)**

*For degree $d \geq 5$, there is no general formula by finitely many radicals to find all roots of every degree $d$ polynomial.*

- **Conclusion:** the $d$ roots have to be found by approximative (=iterative) methods.

**Theorem 3 (Curt McMullen; PhD thesis)**

*For degrees $d > 3$ there is no holomorphic root finding method iterating on $\mathbb{C}$ (one variable) that is generally convergent (i.e. converges to some root on an open dense subset of $\mathbb{C}$ for all polynomials).*

# 3 Several root-finding methods

Numerical analysis has numerous algorithms to find the roots:

1. Newton's method — finds one root at a time (when it does), well understood as a one-dimensional dynamical system
2. Weierstrass method (Durand–Kerner) — finds all $d$ roots simultaneously (when it does), good in practice, seems to converge fast in most cases, but no theory
3. Ehrlich–Aberth-method — similar to Weierstrass' method, but apparently faster: "method of choice" for many
4. Eigenvalue methods — seem to work well for moderate degrees
5. Victor Pan's algorithm: almost best possible theoretical complexity, but unusable in practice
6. many further methods

Our focus today: the first three methods (blue); iteration of complex analytic mappings. — *Dynamical systems in heavy practical use — but very hard to understand*

# 4 Overview

Root-finding methods by Newton, Weierstrass, Ehrlich–Aberth

*Motto: what is the difference between theory and practice?*

Local properties, global questions; do they always converge? almost?

positive practical evidence — BUT failure of global convergence;

I. New theorem: *Weierstrass is not generally convergent*

II. New theorem: *Weierstrass and Ehrlich–Aberth have orbits that are always defined and converge, but not to roots*

III. Conjecture: *Ehrlich–Aberth not generally convergent either*

IV. Open challenges: *Global theory for Weierstrass, Ehrlich–Aberth, and improved (efficient!) Newton methods*

V. Good theory estimates for Newton: quadratic in degree (theoretical bound), log-linear in experiments

VI. Experiments: *Comparison between Newton, Ehrlich–Aberth, and eigenvalue: which one has the advantage?*

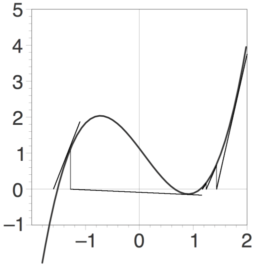
# 5 The Newton Method

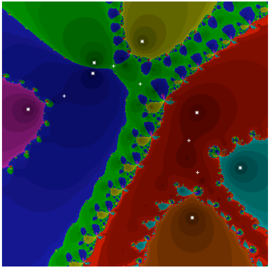
Given differentiable $f\colon \mathbb{R} \to \mathbb{R}$, or $f\colon \mathbb{C} \to \mathbb{C}$, the Newton map is

$$z \mapsto N_f(z) = z - f(z)/f'(z)$$

*Idea:* the best-approximating linear map to $f$ at $z$ is $w \mapsto (z-w)f'(z) + f(z)$, its zero is at $z - f(z)/f'(z)$.

This is good approximation when $w$ is close to $z$; otherwise, iterate . . . and hope!



This finds one zero at a time; how can we go for all roots?

# 6 Derivation of Weierstrass and Ehrlich–Aberth

Let $\mathcal{P}_d$ be the space of monic polynomials of degree d. Given $p(z) = \prod_i (z - \alpha_i) \in \mathcal{P}_d$ and $(z_1, \dots, z_d) \in \mathbb{C}^d$, consider

$$q_i(z) = \frac{p(z)}{\prod_{j \neq i}(z - z_j)} \in \mathbb{C}(z)$$

Heuristic interpretation: if $p(z) = \prod_i (z - \alpha_i)$ and $z_j \approx \alpha_j$, then $q_i(z) \approx z - \alpha_i$.

(Each coordinate thinks that the others must be correct)

Weierstrass (=Durand–Kerner):

$$z_i \mapsto z_i - q_i(z_i)$$

solves $q_i$ as linear polynomial

Ehrlich–Aberth:

$$z_i \mapsto z_i - \frac{q_i(z_i)}{q_i'(z_i)}$$

Newton step for $q_i$

Update all $d$ components in parallel: dynamics in $\mathbb{C}^d$ (Jacobi style)
*Alternative:* use each updated component immediately for the following ones (Gauss–Seidel style) $\rightarrow$ not iteration!

# 7 Relations of Weierstrass to Newton

How to find all $d$ roots of a degree $d$ polynomial $p$ at once?

Given polynomial $p(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_1 z + a_0 = \prod_i (z - \alpha_i)$

From a root vector $(\alpha_1, \ldots, \alpha_d)$, the coefficient vector $(a_1, \ldots, a_d)$ is determined through elementary symmetric functions.
Need to compute the inverse $(a_1, \ldots, a_d) \mapsto (\alpha_1, \ldots, \alpha_d)$.

We have to solve $d$ equations in $d$ variables. Can use Newton's method in $d$ variables and iterate. This iteration equals the *Weierstrass method*.

*Consequence: invariant hyperplane* in which the sum of all components is constant, Weierstrass projects to this hyperplane, hence iteration in $\mathbb{C}^{d-1}$.

# 8 Relations of Ehrlich–Aberth to Newton

Write Newton in one variable, using roots $\alpha_i$

$$z \mapsto z - p(z)/p'(z) = z - \left( \sum_i \frac{1}{z - \alpha_i} \right)^{-1}$$

(roots not known ahead of time, but expression helpful anyway).

*Electrostatic interpretation:* have test charges at positions of the roots $\alpha_i$; the Newton displacement $N_p(z) - z$ is (the inverse of) the electrostatic field at $z$.

For $d$ orbits in parallel with approximation vector $(z_1, \ldots, z_d)$, produce next generation approximation vector $(z'_1, \ldots, z'_d) = EA(z_1, \ldots, z_d)$ via improved Newton step (implicit deflation):

$$z_k \mapsto z_k - q_k(z_k)/q'_k(z_k) = z_k - \left( \sum_i \frac{1}{z_k - \alpha_i} - \sum_{i \neq k} \frac{1}{z_k - z_i} \right)^{-1}$$

Attracting charges at roots, repelling charges at other approximations!

# 9 Local Properties, Global Problems

For all three methods, *points* $z \in \mathbb{C}$ *(for Newton) or [components of] vectors* $(z_1, \ldots, z_d) \in \mathbb{C}^d$ *(for the others) are fixed if and only if one root, resp. all [some] roots, are found*.

Such fixed points are *attracting*: there is a neighborhood in $\mathbb{C}$ resp. $\mathbb{C}^d$ so that all these points converge to the root(s).

The convergence is always very fast: quadratic (number of valid digits doubles in each step); Ehrlich–Aberth is even cubic.

Global properties difficult: where does one have to start? how many iterations required (what is the complexity)?

Previously known problems:

a) symmetry; for real polynomials with non-real roots, real starting vectors, convergence must fail

b) basin boundaries: different root (vectors) have disjoint open basins, $\Longrightarrow$ $\exists$ basin boundaries; may have positive measure!

c) orbits may jump to $\infty$ (Newton) or fail to be defined (points of indeterminacy) (Weierstrass / E–A)

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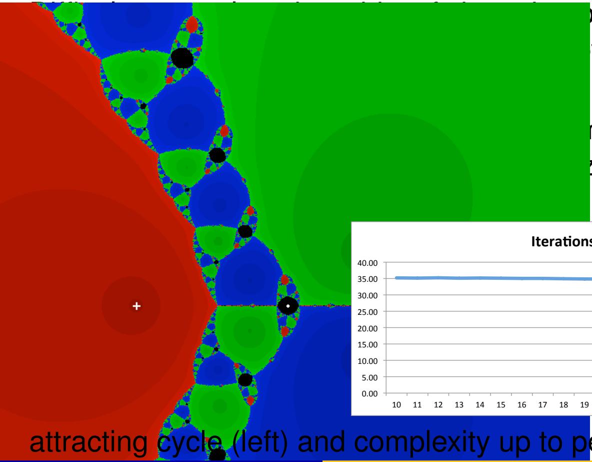
# 10 Empirical evidence: Newton

Newton: for as simple polynomials as $z \mapsto z^3 - 2z + 2$, there are attracting orbits: open sets in $\mathbb{C}$ that converge to periodic cycles but not roots. *Newton's method is not generally convergent!*



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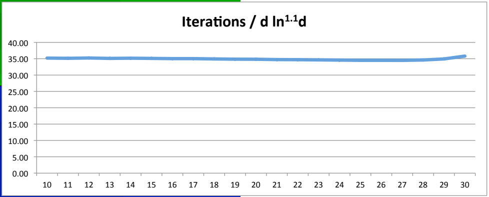



attracting cycle (left) and complexity up to period $2^{30} > 10^9$

# 11 Empirical evidence: Weierstrass & Ehrlich–Aberth

Weierstrass / Ehrlich–Aberth: are "known to converge in all cases, except the obvious cases with symmetry"; have passed the test of time over decades in uncounted experiments. Successful in standard implementations such as MPSolve, degrees up to millions.

Recent (heuristic) improvement by Dario Bini (Pisa) to MPSolve for special (recursive) polynomials (inspired by our results):
Ehrlich–Aberth successful in selected cases for degrees up to billions.

# 12 What can go wrong? Complex manifolds and eigenvalues

Main (known) problem are attracting cycles: periodic orbits for which a neighborhood converges to these, rather than to roots.

**Lemma**: *If $f\colon \mathbb{C}^k \to \mathbb{C}^k$ differentiable with $f(w) = w$ and $Df|_w$ has all eigenvalues in $\mathbb{D}$, then $w$ is attracting.*

For us, $k = 1$ for Newton and $k = d$ for Weierstrass / Ehrlich–Aberth.

If $f$ is Newton, Weierstrass, E-A, then this is what happens at roots. But if $f$ is an iterate, we obtain an attracting cycle. This situation is stable under perturbations:
attracting cycles spoil general convergence.

If $k > 1$ and only $m < k$ eigenvalues are in $\mathbb{D}$, then there is an invariant $m$-dimensional complex manifold (the stable manifold) on which orbits converge to the cycle (tangent to the appropriate eigenvectors).
*Semi-attracting cycles!*

But does this happen? [And can there be Wandering Domains???]

# 13 Parameter space and dimension count

Space of monic complex polynomials of degree $d$ is $\mathbb{C}^d$ (parametrized e.g. by coefficients or by roots).

All three methods respect affine coordinate changes: these lead to conjugate dynamics with same (semi-)attracting dynamics. Hence space of non-equivalent polynomials (parameter space) is complex $d-2$-dimensional.

Easiest case: cubic polynomials, up to conjugation can suppose roots are at 0, 1, $\lambda \in \mathbb{C}$: parameter space is $\mathbb{C}$ for all three methods. *Fundamental case!*

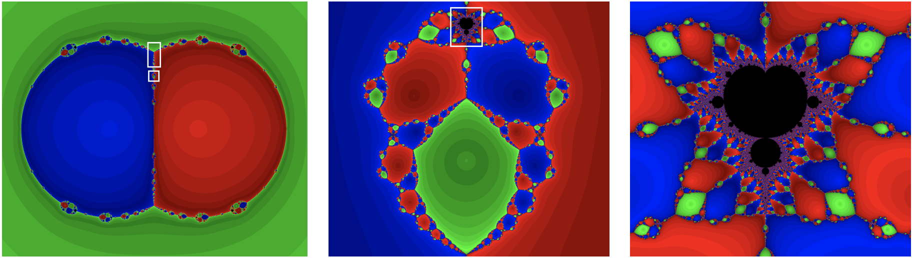
Equivalently, parametrize as $p(z) = z^3 + \lambda z + 1$ or in other ways.

Every periodic point satisfies an algebraic equation in $\lambda$, so all eigenvalues are algebraic functions. Can they all be in $\mathbb{D}$?
$\Longrightarrow$ real algebraic question!

Dimension count: for cubic Newton: one parameter, one eigenvalue:
$\Longrightarrow$ there must (generically) be attracting cycles of all periods!

# 14 Cubic Newton parameter space

**Theorem** [Douady–Hubbard, 1980's]
*The set of parameters $\lambda$ so that the Newton method for $p(z) = z(z-1)(z-\lambda)$ has an attracting cycle of period $n \geq 2$ is contained in finitely many homeomorphic copies of the Mandelbrot set.*



# 15 Cubic Weierstrass

Cubic Weierstrass: one parameter, three eigenvalues (iteration in $\mathbb{C}^3$): generically, expect semi-attracting cycles.

*Reminder:* dynamics projects to invariant $d-1$-dim. hyperplane, so one eigenvalue always zero, two eigenvalues remain.

There is *no a priori reason* why both eigenvalues can or cannot be in $\mathbb{D}$. (Separate question for each period.)

**Conjecture** (Steven Smale): there should be attracting cycles

**Common wisdom** (numerical analysis community): attracting cycles would have been found experimentally (decades ago!) if they existed.

Since both eigenvalues are algebraic functions on the same surface, they must satisfy an algebraic (polynomial) relation.

So question moves from numerical analysis via dynamical systems to algebra; computer algebra brings new tools !

# 16 Dynamical results about Weierstrass method

Theorem A (Reinke–S.–Stoll 2020)

*There is an open set of polynomials $p$ of every degree $d \geq 3$ such that the (partially defined) Weierstrass iteration $W_p \colon \mathbb{C}^d \to \mathbb{C}^d$ associated to $p$ has attracting cycles of period 4. Period 4 is minimal with this property.*

- $\Rightarrow$ Weierstrass's method is not generally convergent!
- Existence of attracting cycles was already conjectured by Smale, has never been observed in practice.
- This work unites numerical analysis, dynamical systems, algebra, and computer algebra (up to current limits!).
- *Math of Computation, February 2023*

Theorem B (Bernhard Reinke, Proc AMS 3/2022)

*For every polynomial $p$ of degree $d \geq 3$ and with distinct roots, there are vectors in $\mathbb{C}^d$ whose orbits under $W_p$ tends to infinity.*

# 17 Proof strategy for Weierstrass attracting cycles

Let $\mathcal{P}_d$ be the space of monic polynomials of degree $d$. We consider

$$\mathcal{Q}_d(n) := \{(p, \underline{z})\colon p \in \mathcal{P}_d \text{ and } \underline{z} \text{ is of period } n \text{ for } W_p\}$$

and

$$M_d(n) := \{(p, \underline{z}, (\mu_1, \ldots, \mu_d))\colon (p, \underline{z}) \in \mathcal{Q}_d(n), \mu_i \text{ eigenvalues of } D(W_p^{\circ n})(\underline{z})\}$$

as algebraic varieties over $\mathcal{P}_d$.

A cycle is attracting iff all $|\mu_i| < 1$, so there are attracting cycles of period $n$ iff the image of $\pi_\mu\colon M_d(n) \to \mathbb{C}^d$ intersects $\mathbb{D}^d$.

With computer algebra, we find explicit equations for $M_d(n)$ and $\overline{\pi_\mu(M_d(n))} \subset \mathbb{C}^d$, and check for intersection with $\mathbb{D}^d$.

# 18 Dimension counts

$$M_d(n) := \{(p, \underline{z}, (\mu_1, \ldots, \mu_d) : (p, \underline{z}) \in \mathcal{Q}_d(n), \mu_i \text{ eigenvalues of } D(W_p^{\circ n})(\underline{z})\}$$

- We expect $\dim M_d(n) = \dim \mathcal{Q}_d(n) = \dim \mathcal{P}_d = d$, but multipliers are invariant under affine congujation, so $\pi_\mu(M_d(n))$ has at least (complex) codimension 2 in $\mathbb{C}^d$.
- The Weierstrass map $W_p$ takes images in the hyperplane of dimension $d - 1$ where sum of coordinates equals sum of roots, so one eigenvalue is always 0.
- The simplest interesting case are cubic polynomials: one parameter dimension, two eigenvalues (other than 0).
- $\overline{\pi_\mu(M_3(n))}$ is a collection of curves in $\mathbb{C}^2$, each defined by a single equation.
- $\Rightarrow$ computations become feasible for Weierstrass.

# 19 Multiplier relations for low periods

- Period 2: general 2-cycles have multipliers $\{(\mu_1, \mu_2) : \mu_1\mu_2 - 2(\mu_1 + \mu_2) + 6 = 0\}$
- Period 3, special case of cycles that have the form $(z_1, z_2, z_3) \mapsto (z_2, z_3, z_1)$ have multipliers $\left\{(\mu_1, \mu_2) = \left(\lambda_1^3, \lambda_2^3\right) : \lambda_1 + \lambda_2 = -3\right\}$

In these cases we can directly see that no such cycles can be attracting.

# 20 Multiplier relations for general 3-cycles

For 3-cycles of general type we have multipliers
$\{(\mu_1, \mu_2) : \mu_1, \mu_2 \text{ roots of } c_2(u)X^2 + c_1(u)X + c_0(u) = 0 \text{ for some } u\}$
where

$$c_0(u) = -9u^{12} - 162u^{11} - 693u^{10} + 1434u^9 + 11958u^8 - 32202u^7 - 182301u^6 + 578742u^5 + 2069910u^4 - 919718u^3 - 3065685u^2 + 892254u + 264295\,,$$

$$c_1(u) = u^{12} + 26u^{11} + 230u^{10} + 693u^9 - 3867u^8 - 5844u^7 + 123074u^6 - 38381u^5 - 1320149u^4 + 420552u^3 + 4310940u^2 - 4206447u + 1442574\,,$$

$$c_2(u) = -9u^{10} - 63u^9 + 301u^8 + 1126u^7 - 7693u^6 - 3641u^5 + 52375u^4 + 13526u^3 - 104463u^2 - 47919u + 20987\,.$$

More computations: *no such 3-cycle can be attracting*.
Heavy computation with state of the art computer algebra systems! — higher periods seem out of reach

# 21 Multiplier relations for transposition 4-cycles

Period 4, special case of 4-cycles of type $(z_1, z_2, z_3) \mapsto (z_4, z_5, z_6) \mapsto (z_2, z_1, z_3)$ we have multipliers $\{(\mu_1, \mu_2) : \mu_1, \mu_2 \text{ roots of } X^2 + c_1 X + c_0 = 0\}$ where $c_0$ and $c_1$ satisfy

$$
\begin{aligned}
34c_0^4c_1^2 + 169c_0^3c_1^3 - 675c_0^2c_1^4 - 2997c_0c_1^5 - 2187c_1^6 + 68c_0^5 \\
+ 984c_0^4c_1 + 3359c_0^3c_1^2 - 19182c_0^2c_1^3 - 88965c_0c_1^4 \\
- 91584c_1^5 + 4254c_0^4 + 29059c_0^3c_1 - 93688c_0^2c_1^2 \\
- 634050c_0c_1^3 - 809379c_1^4 + 76045c_0^3 \\
+60846c_0^2c_1 - 725626c_0c_1^2 - 1171592c_1^3 \\
+ 487003c_0^2 + 4167623c_0c_1 + 8653407c_1^2 \\
+5442895c_0 + 15506760c_1 - 35154225 = 0\,,
\end{aligned}
$$

Double root near $\mu_i \approx -0.6891666 \Rightarrow$ attracting cycle!

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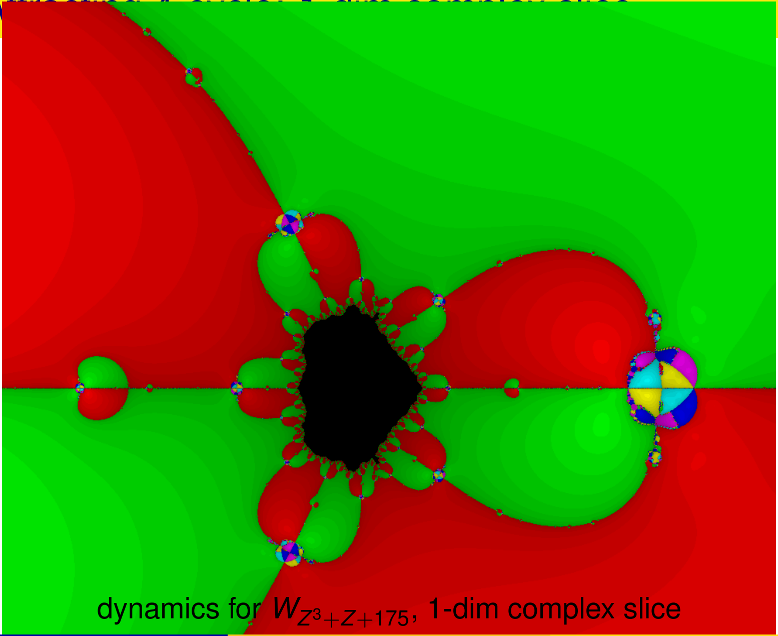
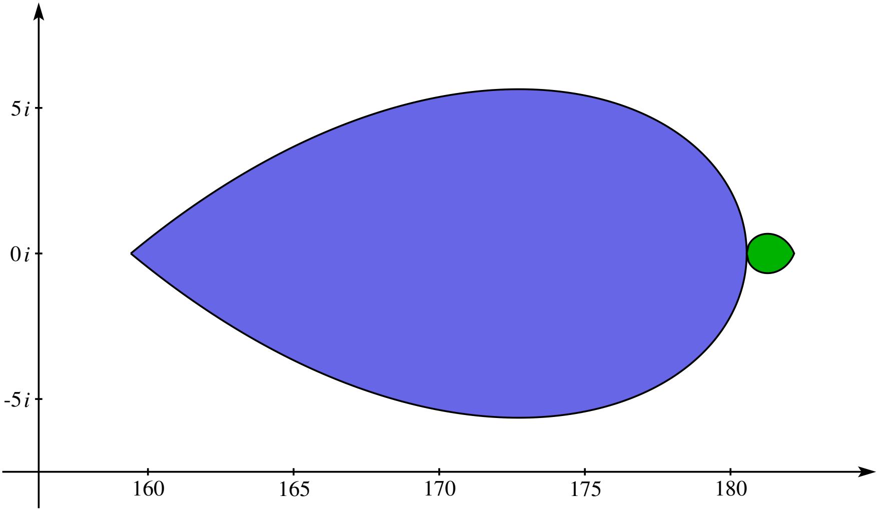
# 22 Attracting 4-cycle: 1-dim complex slice



dynamics for $W_{Z^3+Z+175}$, 1-dim complex slice

# 23 parameter space: $W_{z^3+z+t}$ with attracting 4-cycle



# 24 Next goal: is Ehrlich–Aberth generally convergent?

The question can be asked in the same way; similar methods can be used for Ehrlich–Aberth. Three additional difficulties:

- The Ehrlich–Aberth-map is more complicated than Weierstrass;
- Ehrlich–Aberth does not project to a hyperplane, so need to consider one more non-trivial eigenvalue;
- Ehrlich–Abert is invariant not only under affine maps, but even under all Möbius maps.

The latter seems like a good property, however:

Up to equivalence, there are only three different cubic Ehrlich–Aberth-cases: all roots distinct, or one double/one simple root, or one triple root $\Longrightarrow$ cubic parameter space trivial, so interesting case is degree 4 with one complex degree of freedom: even more complicated.

Seems out of reach of current computer algebra systems :-(

# 25 Is Ehrlich–Aberth generally convergent?

Different approach: mix computer algebra with heavy numerics

Parametrize degree 4 polynomials as $p_\lambda(z) = (z^2 - 1)(z^2 - \lambda)$

A *symmetry 2-cycle* is a point $z \in \mathbb{C}^4$ with $EA(z) = -z$. It has *permutation symmetry* if in addition $-z$ is a permutation of $z$.

**Theorem 4**

*For generic $\lambda$, the map EA has 3 symmetry 2-cycles with permutatiton symmetry, 8 further symmetry 2-cycles, and 1265 general 2-cycles.*

**Theorem 5 (Bernhard Reinke)**

*Cycles with permutation symmetry can never be attracting (any period).*

This excluces the simplest possible cases.

# 26 Symmetry 2-cycles for Ehrlich–Aberth

The 8 symmetry 2-cycles (without permutation) break up into 4 pairs A, B, C, D.

**Theorem 6 (Caspar Kiehn)**

*For the pair A of regular symmetry two-cycles, the four elementary symmetric polynomials of the eigenvalues take the following form (for $\lambda$ or $-\lambda$):*

$$E_1(\lambda) = 80\frac{\lambda}{(\lambda-1)^2}$$

$$E_2(\lambda) = \frac{-9\lambda^4 - 60\lambda^3 + 1850\lambda^2 - 60\lambda - 9}{(\lambda-1)^4}$$

$$E_3(\lambda) = \frac{4\lambda^6 - 456\lambda^5 - 2884\lambda^4 + 12304\lambda^3 - 2884\lambda^2 - 456\lambda + 4}{(\lambda-1)^6}$$

$$E_4(\lambda) = \frac{53\lambda^6 + 290\lambda^5 + 1195\lambda^4 - 11268\lambda^3 + 1195\lambda^2 + 290\lambda + 53}{(\lambda-1)^6}$$

**Corollary:** pair A can never be attracting

# 27 Symmetry 2-cycles for Ehrlich–Aberth

### Theorem 7 (Caspar Kiehn)

*For the pair B of regular symmetry two-cycles, the four elementary symmetric polynomials of the eigenvalues take the following form (for $\lambda$ or $-\lambda$):*

$$E_1(\lambda) = -8\frac{\lambda^4 - 18\lambda^2 + 1}{(\lambda^2 - 1)^2}$$

$$E_2(\lambda) = 2 \cdot \frac{11\lambda^8 - 424\lambda^6 - 7622\lambda^4 - 424\lambda^2 + 11}{(\lambda^2 - 1)^2(\lambda^4 + 10\lambda^2 + 1)}$$

$$E_3(\lambda) = -24 \cdot \frac{\lambda^8 - 104\lambda^6 - 1266\lambda^4 - 104\lambda^2 + 1}{(\lambda^2 - 1)^2(\lambda^4 + 10\lambda^2 + 1)}$$

$$E_4(\lambda) = 2 \cdot \frac{25\lambda^8 - 952\lambda^6 - 8130\lambda^4 - 952\lambda^2 + 25}{(\lambda^2 - 1)^2(\lambda^4 + 10\lambda^2 + 1)} .$$

**Corollary:** pair B can never be attracting either.

The other two pairs are more complicated, currently under investigation.

**Hope:** find attracting 2-cycles among the 1265 general 2-cycles

# 28 Diverging orbits for Weierstrass

For cubic polynomials of the form $p = Z^3 + aZ + b$, $W_p$ projects to the $\mathbb{C}^2$- hyperplane $H = z_1 + z_2 + z_3 = 0$. We can extend $W_p$ to a rational map $\mathbb{P}^2 \to \mathbb{P}^2$

$$\begin{aligned}(z_1 : z_2 : z_0) \longmapsto \big(&(z_1 + 2z_2)(z_1^3 + az_1 z_0^2 + bz_0^3 - z_1 z_2(z_1 + z_2))\\ &:(2z_1 + z_2)(-z_2^3 - az_2 z_0^2 - bz_0^3 + z_1 z_2(z_1 + z_2))\\ &:z_0(z_1 - z_2)(z_1 + 2z_2)(2z_1 + z_2)\big)\ .\end{aligned}$$

The map at the line at infinity $L_\infty$ is independent of $p$, given by

$$\varphi(z) = \Big(z(2 + z)(1 + z - z^2)\Big) \,/\, \Big((1 + 2z)(1 - z - z^2)\Big)\ .$$

It has a periodic point $q_0 \in L_\infty$ that is repelling on $L_\infty$ but has an attracting eigenspace. This gives rise to diverging orbits.

**Theorem B (Bernhard Reinke, Proc AMS 3/2022)**

*For every polynomial p of degree $d \geq 3$ and with distinct roots, there are vectors in $\mathbb{C}^d$ whose orbits under $W_p$ tends to infinity.*
*Analogous result for Ehrlich–Aberth.*

# 29 Gauss–Seidel variants

- So far we updated all coordinates at the same time (“Jacobi”).
- We can also use the new coordinates directly (very natural from numerical perspective).
- After appropriate adjustments, we still have a rational map $\mathbb{C}^d \dashrightarrow \mathbb{C}^d$, but we lose some symmetry.
- For Weierstrass, no invariant hyperplane anymore.

**Theorem C (Reinke 2020)**

*For every polynomial p of degree $d \geq 3$ and with distinct roots, there are vectors in $\mathbb{C}^d$ whose orbits under the Gauss-Seidel variant of the Weierstrass map for p tends to infinity.*

Proof similar to Jacobi case, but have complex plane $E_\infty \subset \mathbb{P}^3$ at infinity instead of complex line.

(In all cases upgrade from degree 3 to $d > 3$ easy.)

# 30 Summary of local results

| Method | param. dim. | dyn. dim. | attr. cycles | diverg. orbits | gen. conv. |
|---|---|---|---|---|---|
| Newton | $d-2$ | 1 | ✓ | ✗ | ✗ |
| Weierstrass | $d-2$ | $d-1$ | ✓ | ✓ | ✗ |
| W. Gauss–Seidel | $d-2$ | $d$ | ? | ✓ | ? |
| Ehrlich–Aberth | $d-3$ | $d$ | ? | ✓ | ? |
| E-A Gauss–Seidel | $d-3$ | $d$ | ? | ? | ? |

Table: Root-finding methods for polynomials of degree $d$ (new results in color)

# 31 Global Results ???

All these are local results: periodic points and their eigenvalues. But what are the global properties of the dynamics?

In particular: where do we need to start the iteration to find the roots?

No global theory for Weierstrass and Ehrlich–Aberth: *they seem to work and nobody knows why*.

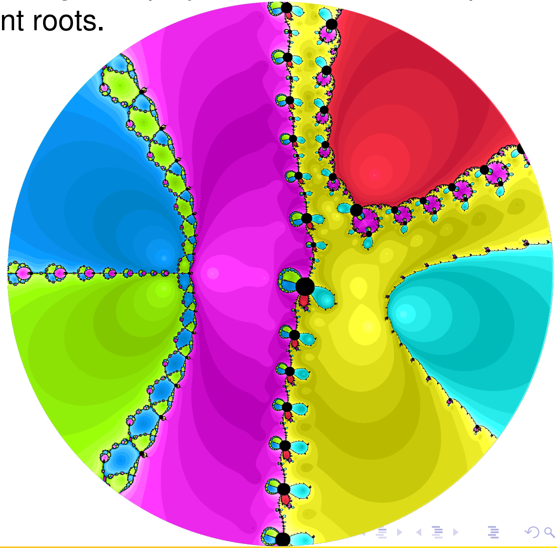
Experience suggests to start with $d$ equidistributed points on large circle. *No nontrivial* convergence results!

However, for Newton there is meanwhile quite a bit of global theory — and *open questions!*

My “mathematical home”: dynamical systems. Interesting interaction with root finding experts (Dario Bini, Victor Pan)

# 32 Global theory for Newton's method

Dynamics of Newton map for degree 6 polynomial. Colors code points in $\mathbb{C}$ that converge to different roots.



Some "old" results:

Let $d$ be the degree of a polynomial.

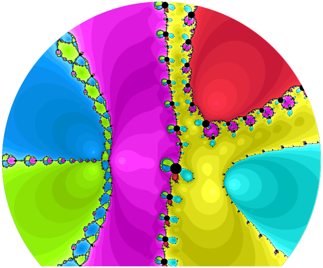
⋆ Dynamics near $\infty$ is repelling: $z \mapsto z(d-1)/z$

⋆ Immediate basin of root: component of basin around the root. Simply connected, unbounded, has one or more accesses to $\infty$ ("channels")

⋆ at least one access is "thick": about $1/d \log d$ of full circle

# 33 Good Starting Points for Newton's Method

**Theorem** (Przytycki, 1980's).
Every immediate basin is simply connected and unbounded, extends to $\infty$ in $k \geq 1$ directions "channels".



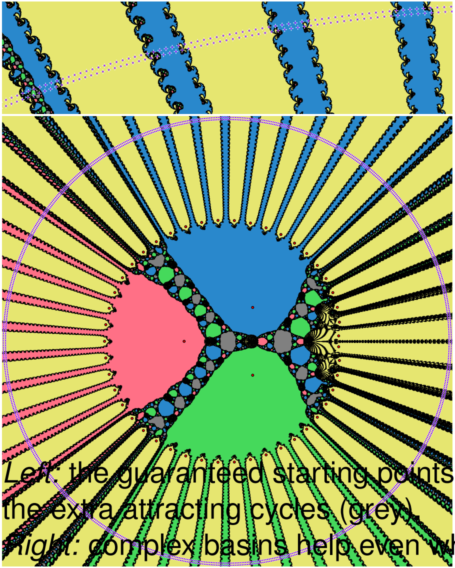
Theorem 8 (Hubbard, S., Sutherland, 2001)

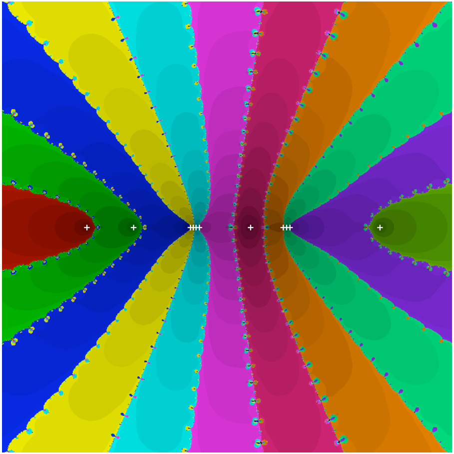
Every root has at least one channel with "thickness" $\pi / \log d$. Hence $1.1 d \log^2 d$ starting points suffice to find all roots.
If all roots are real, then $1.3\, d$ starting points suffice.

Theorem 9 (Bollobás, Lackmann, S., 2011)

With high probability, $O(d(\log \log d)^2)$ starting points suffice.

# 34 Good starting points





*Left:* the guaranteed starting points for a degree 50 polynomial. Note the extra attracting cycles (grey).

*Right:* complex basins help even when all roots are real. Near-multiple roots are easily distinguished

# 35 Theory on number of iterations for Newton

**Theorem 10 (S.; *Nonlinearity*, 2023)**

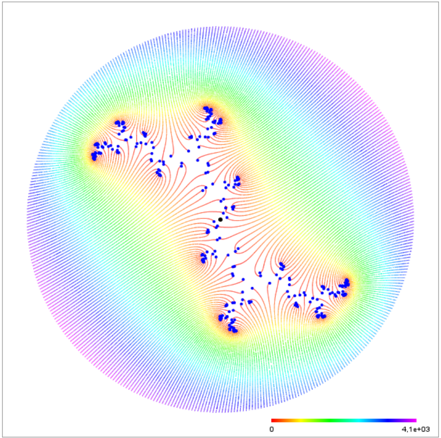
Expected complexity for Newton to find all roots for degree $d$ polynomial is at most $O(d^2 \log^4 d)$.

More precisely, to achieve complecity $\varepsilon > 0$, expected complexity (in terms of number of Newton iterations) is $O(d^2 \log^4 d + d \log|\log \varepsilon|)$.

Advantage: This is the only known root finder that works in practice and has explicit complexity bounds!

Disadvantage: this bound is quadratic in $d$, hence of little interest to numerics people ("we know that our root finders are log-linear in practice!")

# 36 The $d^2$ bound is best possible — and wasteful!




All orbits have to start in area of controlled dynamics, away from disk containing roots.
On this domain $N_p(z) \approx \frac{d-1}{d} z$.
To move a factor of $r$ towards $\mathbb{D}$, each orbit needs $\approx d \log r$ iterations.
Need at least $d$ orbits for $d$ roots: complexity $\geq \Omega(d^2)$.
*Interesting:* upper and lower bounds for theory coincide up to log-factors.

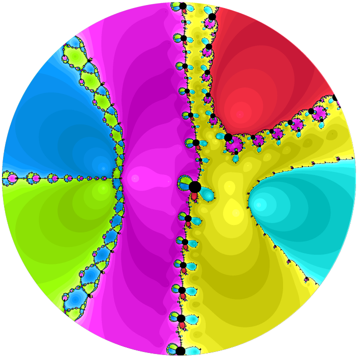
# 37 First step in global theory: *R*-central orbits

First major problem of Newton: dynamics chaotic on Julia set, so orbits can be unbounded.

⋆ **Definition:** a Newton orbit is called *R-central* if it stays within disk of radius $R$

⋆ **Theorem:** *use* ***twice as many*** *starting points in guaranteed universal set of starting points. Then each root has at least one R-central point in its immediate basin, for* $R \le 7$

**Idea:** controlled dynamics on channels near $\infty$: displacement from $z$ to $N_p(z)$ has hyperbolic distance (within immediate basin) at most $1/\log d$ in center of channel



$\Longrightarrow$ twice enlarged set of starting points will find "central sub-channel" of points with hyperbolic distance at most $2/\log d$

$\Longrightarrow$ all orbits must reach disk containing roots, cannot escape too far

# 38 Bounding number of iterations: area per iteration

Overall idea: each Newton iteration requires a certain area, these areas are essentially disjoint, orbits stay in uniform disk: bounded area implies bounded number of iterations.

$\star$ expected mutual displacement between roots is $O(1/d)$
$\star$ **Lemma:** if $|N(z) - z| < 1/d^2$, then $|z - \alpha| < 1/2d$ for nearest root $\alpha$ and quadratic convergence in at most $O(\log|\log \varepsilon|)$ iterations
$\star$ until that time, Newton orbit has $|N(z) - z| \geq 1/d^2$ but hyperbolic displacement $< 2 \log d$ in immediate basin
$\star$ upper bound for hyperbolic displacement and lower bound for Euclidean displacement $\implies$ distance to boundary of imme- diate basin at most $1/2d^2 \log d$ $\implies$ can connect $z$ to $N(z)$ by "thick curve" of width $1/d^2 \log d$; needs area at least length x width $\geq (1/d^2) \times (1/d^2 \log d) = 1/d^4 \log d$
$\star$ but most orbit points are further away from roots, hence greater displacement, greater area: average area $1/d^2 \log d$, hence $d^2 \log d$ iterations required

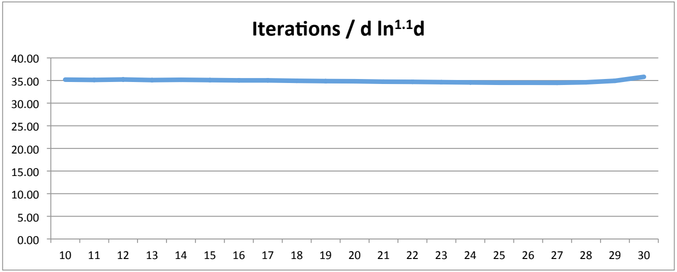
# 39 Beating the quadratic complexity estimate

Situation so far:

⋆ upper complexity bound $O(d^2 \log^4 d$ iterations shown)

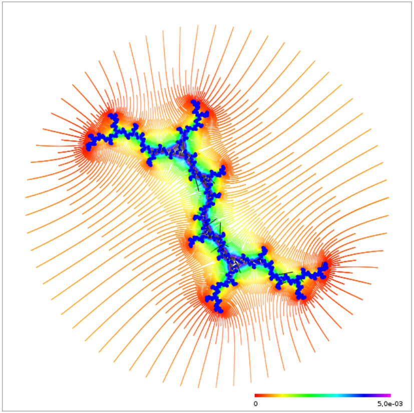
⋆ for starting points on circle complexity is at least quadratic

but have efficient heuristic experiments by high school students:




complexity for finding all roots of polynomials of degrees up to $2^{30} \approx 10^9$

# 40 The Iterated Refinement Method




Natural heuristics: in area where dynamics “parallel”, use only few orbits and refine as dynamics becomes non-linear.

**Danger:** no longer clear that all roots are found. If roots missing, no obvious place for refining Newton dynamics.

“Experimental” algorithm (heuristic as all!)

Highly successful experiments: again Robin Stoll, now young university student — and Marvin Randig, another high school student. Up to degrees $2^{30} > 10^9$ on a student laptop!

Global theory missing! Need more progress!

# 41 Recent progress

Nicolae Mihalache, François Vigneron (2024):

*How to split a tera-polynomial*: adapted Newton's method to factor polynomials of degree $2^{40} > 10^{12}$. (No theory yet.)

Victor Pan, S. (work in progress):

combination between global "subdivision of squares" algorithm and local adaptation of Newton theory: new efficient algorithm in development, hopefully with theoretical support.

# 42 Problems and projects

*Surprising:* classical and fundamental problem, so wide open!

⋆ Challenge: show that Ehrlich–Aberth is not generally convergent

⋆ Challenge: develop theory and good complexity for *Iterated Refinement Newton Method*

⋆ Challenge: can Weierstrass have attracting cycles when starting points are on regular $d$-gon?

⋆ Challenge: can there be any generally convergent root finding method that finds roots in parallel?

⋆ Task: make systematic comparison (in practice) between Newton, Ehrlich–Aberth, and eigenvalue methods: which method has an advantage for which kind of polynomials?

# 43 Conclusion

⋆ root finding is a classical and important problem

⋆ there are various methods that work well in practice (at least for moderate degrees), but GAP between theory and practice

⋆ Weierstrass and Ehrlich–Aberth: basis of standard software package, have very good reputation, but little theory

⋆ failure of general converge (Weierstrass) and diverging orbits are two new phenomena in global dynamics

⋆ Newton used to have poor reputation (except locally), but meanwhile unique good combination of theory and practice

⋆ ... most efficient "iterated refinement" and "Mihalache/Vigneras"-implementations not yet supported by theory

⋆ these root finding methods give rise to most interesting dynamical systems

⋆ many questions on all topics still wide open — new results required!

Thank you very much!

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