## Key definitions of Lecture 1:

- $t(F,G) := \mathbb{P}_{\substack{\text{random } f: \\ V(F)\to V(G)}} \left[ \forall xy \in E(F) \;\; f(x) f(y) \in E(G) \right]$

  (<u>homomorphism density</u> of F in G)

- $(G_n)$ <u>converges</u> to $\phi : \mathcal{G} \to \mathbb{R}$ if

  $$\forall F \quad \lim_{n\to\infty} t(F, G_n) = \phi(F)$$

- $LIM := \{ \text{such } \phi \} \subseteq [0,1]^{\mathcal{G}}$

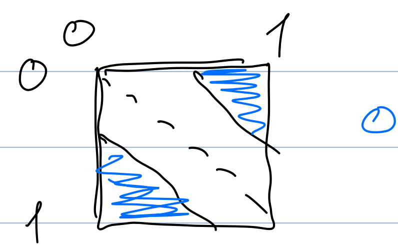
## 4 Graphon

<u>Kernel</u>: measurable bounded fn $W: [0,1]^2 \to \mathbb{R}$ which is <u>symmetric</u> $(W(x,y) = W(y,x))$

<u>graphon</u>: kernel which is $[0,1]$-valued

$\mathcal{W} := \{ \text{kernels} \}$

$\mathcal{W}_0 := \{ \text{graphons} \}$




<u>Ex</u> (of graphons) constant-$p$ ✓, $xy$ ✓, $\mathbb{1}_{|x-y|\geq \frac{1}{2}}$ ✓

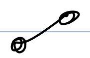
Non-examples: $x+y$ ✗ $\max\{x-y, 0\}$ ✗

(Hom) density of $F \in \mathcal{F}_k := \{\text{Graphs on } \underbrace{[k]}_{\{1,\dots,k\}}\}$

$$\left[\, |\mathcal{F}_k| = 2^{\binom{k}{2}} \,\right]$$

$$t(F,W) := \int_{[0,1]^k} dx_1 \dots dx_k \prod_{ij \in E(F)} W(x_i, x_j)$$

$$t(K_2, W) = \int_{[0,1]^2} W(x,y)\, dx\, dy$$



Thm 4.1 (Lovász-Szegedy '06)

$$LIM = \{ t(-,W) : W \in \mathcal{W}_0 \}$$

"$\supseteq$": $\phi_W(-) := t(-,W)$:

- normalised $\left( \phi_W(K_1) = \int 1 = 1 \right)$
- multiplicative $\left( \phi_W(FH) = \phi_W(F)\, \phi_W(H) \right)$

$$\phi_W^{\uparrow}(F) = \sum_{\substack{F' : V(F') = V(F) \\ E(F') \supseteq E(F)}} (-1)^{e(F') - e(F)} \cdot \int \prod_{ij \in E(F')} W(x_i, x_j)$$

$$= \underbrace{\int_{[0,1]^k} dx_1 \dots dx_k \prod_{ij\in E(F)} W(x_i,x_j) \prod_{ij\in E(F)} \big(1-W(x_i,x_j)\big)}_{\shortparallel} \geq 0$$

$$t_{ind}(F, W)$$

- pick $x_1, \dots, x_n \in [0,1]$
- connect $i$ to $j$ with Prob $W(x_i,x_j)$ (all independent)

Thm 3.2 $\Rightarrow$ $\phi_W \in LIM$.

<u>Random sample</u> $G(n, W)$ equivalent

- rnd graph in $\mathcal{F}_n$ (ie on $[n]$)
- $\mathbb{P}[G(n,W) = F] = t_{ind}(F)$

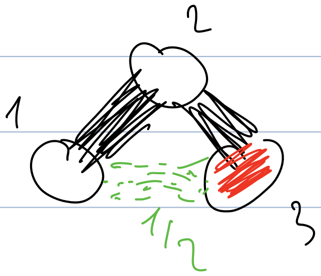
$$\mathbb{E}\Big(t_{ind}\big(F, G(n,W)\big)\Big) = t_{ind}(F, W)$$
$$v(F) = k$$

Concentration: with prob 1, if $G_n \sim G(n,W)$, then $(G_n) \to W$. $\square$

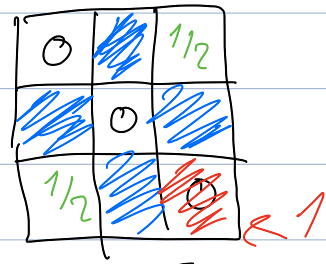
$(G_n)$ <u>converges</u> to $W$: $\forall F$ $t(F, G_n) \to t(F,W)$

<u>Ex</u> $(K_n) \to 1$, $(\overline{K_n}) \to 0$,

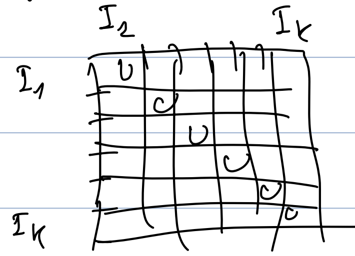
$(K_{n/2,n/2}) \to$ 




$\longrightarrow$




$G(1), G(2), G(3) \longrightarrow$




$W_G$

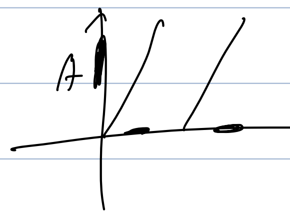
$v(G) = k$

⚠ $\forall$ measure-preserving $\psi : [0,1] \to [0,1]$

the <u>pull-back</u> $W^{\psi}(x,y) := W(\psi(x), \psi(y))$

satisfies $t(F, W^{\psi}) = t(F, W)$

<u>Ex</u> $\psi : x \mapsto 2x \mod 1$



$W^{\psi}$ : $\begin{bmatrix} W & W \\ W & W \end{bmatrix}$

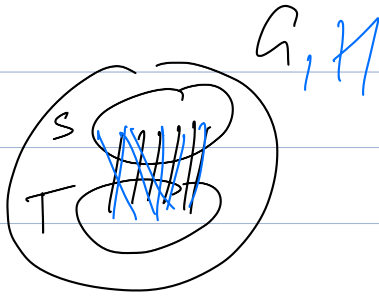
## <u>Borgs-Chayes-Lovász '10</u>

<u>Cut-norm</u> of kernel $W$ is

$$\|W\|_{\square} := \sup_{S,T \subseteq [0,1]} \left| \int_{S \times T} W(x,y)\,dx\,dy \right|$$

$V(H) = V(G) = [n]$

$$\|W_G - W_H\|_\square \sim \frac{1}{n^2} \sup_{T, S \subseteq [n]} |e_G(S,T) - e_H(S,T)|$$




Cut distance: $\delta_\square(W, U) = \inf_{\text{m.p. } \varphi} \|W - U^\varphi\|_\square$

conv in cut dist. $\iff$ hom. dens. conv.

+ compactness

LDP for $G(n,p)$ uses $\mathcal{W}_0$

(Chatterjee-Varadhan '11)

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