ICMU Mini-Course "Limits of Discrete Structures"

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Recommended reading:
- László Lovász *"Large networks and graph limits"* (link)
- Yufei Zhao *"Graph Theory and Additive Combinatorics"* (link)

Notation/Terminology

$[n] := \{1, \ldots, n\}$

Indicator function $\mathbb{1}_A(x) := \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$

$(n)_k := n(n-1)\ldots(n-k+1)$

$\mathcal{F}_k := \{\text{graphs on } [k]\}$

$F \cong G$: F and G are isomorphic graphs

$\mathcal{G} := \{$ finite graphs up to isomorphism $\}$

Order $v(G) = |V(G)|$

Size $e(G) = |E(G)|$

$\overline{G}$: complement of G

P_n, C_n, K_n: resp. path, cycle, clique on n vertices

m-blowup $G(m)$: make m 'clones' of each vertex

$FG := F \sqcup G$ (disjoint union) for $F, G \in \mathcal{G}$

Homomorphism $f : F \to G$*:* map $f : V(F) \to V(G)$ s.t. $f(E(F)) \subseteq E(G)$

$\hom(F, G) := |\{f : F \to G\}|$

$\mathrm{inj}(F, G) := |\{$ injective homs $f : F \to G\}|$

$\mathrm{ind}(F, G) := |\{$ injective homs $f : F \to G$ st $\forall xy \in E(ØF)\ \ f(x)f(y) \in E(ØG)\}|$,

Graph parameter $f : \mathcal{G} \to \mathbb{R}$

Upper Möbius inverse:

$$f^{\uparrow}(F) := \sum_{F' \in \mathcal{F}_k : E(F') \supseteq E(F)} (-1)^{e(F')-e(F)} f(F'), \quad F \in \mathcal{F}_k$$

Let $v(F) = k$ and $v(G) = n$. *Homomorphism densities:*

$$\begin{aligned} t(F, G) &:= \hom(F, G)/n^k \\ t_{\mathrm{ind}}(F, G) &:= \mathrm{ind}(F, G)/(n)_k \\ t_{\mathrm{inj}}(F, G) &:= \mathrm{inj}(F, G)/(n)_k \end{aligned}$$

$\phi : \mathcal{G} \to \mathbb{R}$ is *normalised* if $\phi(K_1) = 1$ and *multiplicative* if $\phi(FG) = \phi(F)\phi(G)$ for all $F, G \in \mathcal{G}$

(G_n) *convergent* to $\phi : \mathcal{G} \to \mathbb{R}$ (denoted $(G_n) \to \phi$) if $\lim_{n\to\infty} t(F, G_n) = \phi(F)$ for all $F \in \mathcal{G}$

LIM $:= \{\phi : \mathcal{G} \to [0,1] : \exists (G_n) \to \phi\} \subseteq [0,1]^{\mathcal{G}}$
(G_n) is *p-quasirandom* if $\lim_{n\to\infty} t(F, G_n) = p^{e(F)}$ for all $F \in \mathcal{G}$
Sampling from $\phi \in$ LIM*:* distribution $\mathbb{G}(n,\phi)$ on $\mathcal{F}_n$ where the probability of $F \in \mathcal{F}_n$ is $\phi^{\uparrow}(F)$
Kernel: measurable bounded function $W : [0,1]^2 \to \mathbb{R}$ which is *symmetric* (i.e. $W(x,y) = W(y,x)$)
Graphon: $[0,1]$-valued kernel (i.e. $0 \le W \le 1$)
$\mathcal{W} := \{\text{kernels}\}$
$\mathcal{W}_0 := \{\text{graphons}\}$
$t(F,W) := \int_{[0,1]^k} \prod_{ij\in E} W(x_i,x_j) dx_1 \dots dx_k$, for $F \in \mathcal{F}_k$ and $W \in \mathcal{W}$
Random sample $\mathbb{G}(n,W)$: take uniform $x_1,\dots,x_n \in [0,1]$ and then connect $ij \in \binom{[n]}{2}$ with probability $W(x_i,x_j)$ (all choices independent)
For $G \in \mathcal{F}_k$, define

$$W_G(x,y) := \begin{cases} 1, & x \in [\frac{i-1}{k}, \frac{i}{k}), y \in [\frac{j-1}{k}, \frac{j}{k}), ij \in E(G) \\ 0, & \text{otherwise} \end{cases}$$

Cut norm $\|W\|_\square := \sup_{S,T} \left| \int_{S\times T} W \right|$
$W^\phi(x,y) := W(\phi(x),\phi(y))$
Cut distance $\delta_\square(U,W) := \inf\{\|U - W^\tau\|_\square : \text{measure-preserving } \tau : [0,1] \to [0,1]\}$

Homework for Lecture 1

Problem 1 Given graphs G_1 and G_2, find a graph G such that for every F the following holds:

$$\hom(F,G) = \hom(F,G_1)\ \hom(F,G_2).$$

Problem 2 Let $\mathrm{surj}(F,G)$ be the number of homomorphisms f from F to G such that $f : V(F) \to V(G)$ is surjective. Fix a graph F. Show that each of the functions $\hom(F,\cdot)$ and $\mathrm{surj}(F,\cdot)$ determines the other.

Problem 3 Let $0 < p < 1$. Prove that there is no graph G such that $t(K_2,G) = p$ and $t(C_4,G) = p^4$. [Hint: you can use Chung-Graham-Wilson's theorem.]

Problem 4 Recall that LIM consists of those functions $\phi : \mathcal{G} \to [0,1]$ for which there is a sequence of graphs (G_n) such that for every $F \in \mathcal{G}$ we have $\phi(F) = \lim_{n\to\infty} t(F,G_n)$. Prove that the set LIM is a closed subset of the product space $[0,1]^{\mathcal{G}}$.

Problem 5 Show that

$$\|W\|_\square := \left| \int_{[0,1]^2} \sup_{S,T\subseteq[0,1]} W(x,y)\,\mathrm{d}x\,\mathrm{d}y \right|$$

is a norm on the vector space $\mathcal{W}$ of kernels where a.e. equal kernels are identified.