On discrete, continuous and arithmetic aspects of Fourier uncertainty

Alex Iosevich

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Finite Signals and Discrete Fourier transform

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- Fourier Inversion says that we can recover the signal by using the Fourier inversion:

$$f(x) = N^{-\frac{d}{2}} \sum_{m \in \mathbb{Z}_N^d} \chi(x \cdot m) \widehat{f}(m).$$

Exact recovery problem

- The basic question is, can we recover f **exactly** from its discrete Fourier transforms if

$$\left\{ \widehat{f}(m) : m \in S \right\}$$

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are unobserved (or missing due to noise, other interference, or security), for some $S \subset \mathbb{Z}_N^d$?

- The answer turns out to be $\boxed{YES}$ if f is supported in $E \subset \mathbb{Z}_N^d$, and

$$|E| \cdot |S| < \frac{N^d}{2},$$

with the main tool being the Fourier Uncertainty Principle.

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- (Plancherel)

$$\sum_{m \in \mathbb{Z}_N^d} |\widehat{f}(m)|^2 = \sum_{x \in \mathbb{Z}_N^d} |f(x)|^2.$$

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- $$= \sum_{y \in \mathbb{Z}_N^d} f(y) N^{-d} \sum_{m \in \mathbb{Z}_N^d} \chi((x-y) \cdot m) = f(x)$$

by orthogonality.

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A few simple calculations: the paraboloid

- Let N be an odd prime and define

$$P = \{x \in \mathbb{Z}_N^d : x_d = x_1^2 + \cdots + x_{d-1}^2\}.$$

We have

$$\widehat{1}_P(m) = N^{-\frac{d}{2}} \sum_{y \in \mathbb{Z}_N^{d-1}} \chi(-y \cdot m' + ||y|| m_d),$$

where

$$||y|| = y_1^2 + y_2^2 + \cdots + y_{d-1}^2.$$

Paraboloid (continued)

- Suppose that $m_d = 0$ and $m' \neq \mathbf{0}$. Then

$$\widehat{1}_P(m', 0) = N^{-\frac{d}{2}} \sum_{y \in \mathbb{Z}_N^{d-1}} \chi(-y \cdot m) = 0.$$

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- $$g(a) = \sum_{t \in \mathbb{Z}_N} \chi(at^2), \text{ the classical Gauss sum.}$$

Gauss sum estimation

- Suppose that N is an odd prime and $a \neq 0$. We have

$$|g(a)|^2 = \sum_{t,s} \chi(a(t^2 - s^2)) = \sum_{t,s} \chi(ats)$$

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- It is not difficult to see that $n(0) = 2N - 1$ and $N - 1$ otherwise, so

$$|g(a)|^2 = 2N - 1 + (N-1) \sum_{u \neq 0} \chi(au)$$

$$= N + (N-1) \sum_u \chi(au) = N.$$

Back to the paraboloid

- It follows that if $a \neq 0$,
$$|g(a)| = \sqrt{N}.$$

Going back to the paraboloid and N is an odd prime, we see that if $m' = \mathbf{0}$, $m_d \neq 0$,

$$|\widehat{1}_M(0, \ldots, 0, m_d)| = N^{-\frac{d}{2}} \sum_{m \in \mathbb{Z}_N^{d-1}} \chi(m_d||y||)$$

$$= N^{-\frac{d}{2}} (\sqrt{N})^{d-1} = N^{-\frac{1}{2}}.$$

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- If $m_d \neq 0$ and $m' \neq (0, \ldots, 0)$, we can complete the square and obtain the same bound, i.e

$$|\widehat{1}_P(m)| = N^{-\frac{1}{2}}.$$

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- Suppose that $m \neq \mathbf{0}$. We have

$$\widehat{1}_S(m) = N^{-\frac{d}{2}} \sum_x \chi(-x \cdot m) N^{-1} \sum_{s \neq 0} \chi(s(||x|| - 1)).$$

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- Since

$$sx_j^2 - x_j m_j = s(x_j^2 - x_j m_j / s) = s(x_j - m_j/2s)^2 - m_j^2/4s^2),$$

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- $$N^{-\frac{d}{2}-1} \sum_{s \neq 0} \sum_{x \in \mathbb{Z}_N^d} \chi(s||x||)\chi(-s)\chi(-||m||/4s).$$

The sphere (continued)

- Using the Gauss sum identity we obtain a few minutes ago, the expression above equals

$$N^{-1}\sum_{s\neq 0}\gamma^d(s)\chi(-s-||m||/4s),$$

where

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- The "innocent" looking expression above is a twisted Kloosterman sum. Its modulus is bounded by $2\sqrt{N}$. The proof of this fact is very sophisticated and uses highly non-trivial number theory.
- In conclusion, if $m\neq \mathbf{0}$,

$$|\widehat{1}_S(m)|\leq CN^{-\frac{1}{2}}.$$

The square root law

- In both the case of the sphere and the paraboloid, we established an estimate of the form

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- An interesting situation arises if we ask whether such estimate can ever hold in a non-field setting. The is where we now (briefly) turn our attention.

Zero divisors are problematic

Theorem

(A.I., B. Murphy and J. Pakianathan (2014)) Let R_i be a sequence of finite rings (not necessarily commutative) such that $|R_i|$ is odd and $|R_i| \to \infty$ as $i \to \infty$. Suppose that

$$\left| \sum_{uv=1} \chi(u,v) \right| \leq C|R_i^*|^{\frac{1}{2}},$$

where χ is a non-trivial character on R_i, and R_i^* is the ring of units of R_i.

Then R_is are eventually finite fields.

Zero divisors are problematic (general formulation)

Theorem

(N. Kingsbury (2024)) Let $f(X_1, \ldots, X_{d-1})$ be a polynomial in $Z[X_1, \ldots, X_{d-1}]$. Let $V_f(R)$ denote the solution set to

$$X_d = f(X_1, \ldots, X_{d-1})$$

over a finite ring R.

Suppose a sequence of finite rings $\{R_i\}$ has the property that Fourier transforms over $V(R_i)$ satisfy square root cancellation (for some fixed constant).

Then all but finitely many of the rings are fields or matrix rings of small dimension relative to d.

From Fourier decay to additive energy

- Suppose that S satisfies

$$|\widehat{1}_S(m)| \leq C_{Fourier} N^{-\frac{d}{2}} \cdot |S|^{\frac{1}{2}} \text{ for } m \neq \mathbf{0}.$$

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- We have $\sum_m |\widehat{1}_S(m)|^4 =$

$$= N^{-2d} \sum_{x,y,x',y} \sum_m \chi(m \cdot (x+y-x'-y'))1_S(x)1_S(y)1_S(x')1_S(y')$$

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$$\Lambda(S) = |\{(x, y, x', y') \in S^4 : x + y = x' + y'\}| = N^d \sum_m |\widehat{1}_S(m)|^4.$$

From Fourier decay to additive energy (continued)

- By assumption, the right-hand side is bounded by

$$N^d \cdot C_{Fourier}^2 \cdot N^{-d} \cdot |S| \cdot \sum_m |\widehat{1}_S(m)|^2.$$

From Fourier decay to additive energy (continued)

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- By Plancherel, this expression equals

$$C_{Fourier}^2 \cdot |S|^2,$$

from which we conclude that

$$\frac{\Lambda(S)}{|S|^2} \leq C_{Fourier}^2.$$

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- $$= N^{-\frac{d}{2}} \sum_{m \notin S} \chi(x \cdot m)\widehat{1}_E(m) + N^{-\frac{d}{2}} \sum_{m \in S} \chi(x \cdot m)\widehat{1}_E(m)$$

An elementary point of view: direct estimation

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- Since we know nothing about S, the best we can do is assume that the quantity above is small.

An elementary point of view: rounding

- If
$$N^{-d}|E||S| < \frac{1}{2},$$
we can take the modulus of $I(x)$ and round it up to 1 if it is $\geq \frac{1}{2}$, and round it down to 0 otherwise.

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- This gives us **exact recovery** using a simple and direct algorithm (to be henceforth referred to as the Direct Rounding Algorithm (DRA)) if

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- But what happens if we consider general signals?

Matolcsi-Szucks/ Donoho-Stark point of view

- Let $h : \mathbb{Z}_N^d \to \mathbb{C}$. Then the classical Uncertainty Principle says that

$$|supp(h)| \cdot |supp(\hat{h})| \geq N^d.$$

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- If f cannot be recovered uniquely, then there exists a signal $g : \mathbb{Z}_N^d \to \mathbb{C}$ such that g also has $|supp(f)|$ non-zero entries,

$$\widehat{f}(m) = \widehat{g}(m) \text{ for } m \notin S,$$

and f is not identically equal to g.

Uncertainty Principle → Unique Recovery

- Let $h = f - g$. It is clear that $\widehat{h}$ has at most $|S|$ non-zero entries, and h has at most $2|supp(f)|$ non-zero entries.

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Uncertainty Principle $\rightarrow$ Unique Recovery

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- By the Uncertainty Principle, we must have

$$|supp(f)| \cdot |S| \geq \frac{N^d}{2}.$$

- Therefore, if we assume that

$$|supp(f)| \cdot |S| < \frac{N^d}{2},$$

we must have $h = 0$, and hence the recovery is *unique*.

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- We are going to see that in the presence of non-trivial restriction estimates, we can do much better. We are also going to see that non-trivial restriction estimates "typically" hold.

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$$|h(x)| \leq N^{-\frac{d}{2}} \cdot |S| \cdot N^{-\frac{d}{2}} \cdot \sum_{x \in \mathbb{Z}_N^d} |h(x)|.$$

- Summing both sides over $x \in E$ and cancelling the L^1 norms of h on both sides, we obtain
$$|E| \cdot |S| \geq N^d.$$

Additive energy uncertainty principle

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Definition (Additive energy)

Let $A \subset \mathbb{Z}_N^d$. The **additive energy** of A, denoted by $\Lambda(A)$, is defined as follows:

$$\Lambda(A) = \left| \left\{ (x_1, x_2, x_3, x_4) \in A^4 : x_1 + x_2 = x_3 + x_4 \right\} \right| .$$

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- This quantity measures the extent to which a given set is arithmetically closed.

Additive energy uncertainty principle

Theorem (Additive Energy Uncertainty Principle)

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ with support in E and supp$(\widehat{f}) = S$. Then for any $\alpha \in [0,1]$,

$$\left(|E| \max_{U \subset S} \frac{\Lambda(U)}{|U|^2} \right)^{\alpha} \cdot \left(|S| \max_{F \subset E} \frac{\Lambda(F)}{|F|^2} \right)^{1-\alpha} \geq N^d.$$

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- Since $|\Lambda(U)| \leq |U|^3$, the results above recover the classical uncertainty principle.
- If $|\Lambda(F)| = o(|U|^3)$ for all $F \subset E$, and/or if $|\Lambda(U)| = o(|U|^3)$ for all $U \subset \Sigma$, which holds in the generic case, we get an improved uncertainty principle.

A familiar example - the circle

- Suppose that N is an odd prime and $d = 2$. Let

$$S = \left\{ m \in \mathbb{Z}_N^2 : m_1^2 + m_2^2 = 1 \right\}.$$

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- Since N is prime, there are more algebraic ways of addressing uncertainty in this setting as we shall eventually see.

Restriction theory enters the picture

- We say that $S \subset \mathbb{Z}_N^d$ satisfies the (p,q) restriction estimate $(1 \le p \le q)$ with uniform constant $C_{p,q} > 0$ if for any function $f : \mathbb{Z}_N^d \to \mathbb{C}$,

$$\left(\frac{1}{|S|} \sum_{m \in S} |\widehat{f}(m)|^q \right)^{\frac{1}{q}} \le C_{p,q} N^{-\frac{d}{2}} \left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^p \right)^{\frac{1}{p}}.$$

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- We shall need the following "universal" restriction theorem.

Theorem

(A.I. and A. Mayeli) Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ and let S be a subset of $\mathbb{Z}_N^d$. Then

$$\left(\frac{1}{|S|} \sum_{m \in S} |\widehat{f}(m)|^2 \right)^{\frac{1}{2}} \leq \left(\frac{|S|}{N^{\frac{d}{2}}} \right)^{-\frac{1}{2}} \cdot \left(\max_{U \subset S} \frac{\Lambda(U)}{|U|^2} \right)^{\frac{1}{4}} \cdot N^{-\frac{d}{2}} \cdot \left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^{\frac{4}{3}} \right)^{\frac{3}{4}}.$$

From restriction directly to uncertainty

- Before proving the universal restricion theorem, we are going to develop a simple mechanism for going directly from restriction to uncertainty, where the more non-trivial the restriction estimate becomes, the better uncertainty principle we obtain. More eleborate versions of this approach will be developed a bit later.

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Theorem (Uncertainty Principle via Restriction Theory – A.I. & A.Mayeli, 2023)

Suppose that $f, \widehat{f} : \mathbb{Z}_N^d \to \mathbb{C}$, with f supported in $E \subset \mathbb{Z}_N^d$, and $\widehat{f}$ supported in $S \subset \mathbb{Z}_N^d$. Suppose S satisfies the (p, q) restriction estimate with norm $C_{p,q}$. Then

$$|E|^{\frac{1}{p}} \cdot |S| \geq \frac{N^d}{C_{p,q}}.$$

Proof of Uncertainty via Restriction

- Suppose that f is supported in a set E, and $\widehat{f}$ is supported in a set S. Then by the Fourier Inversion Formula and the support condition,

$$f(x) = N^{-\frac{d}{2}} \sum_{m \in \mathbb{Z}_N^d} \chi(x \cdot m) \widehat{f}(m) = N^{-\frac{d}{2}} \sum_{m \in S} \chi(x \cdot m) \widehat{f}(m).$$

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- By Holder's inequality,

$$|f(x)| \leq N^{-\frac{d}{2}} \cdot |S| \cdot \left(\frac{1}{|S|} \sum_{m \in S} |\widehat{f}(m)|^q \right)^{\frac{1}{q}}.$$

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- By the restriction bound assumption, this expression is bounded by

$$|S| \cdot C_{p,q} \cdot N^{-d} \cdot \left(\sum_{x \in \mathbb{Z}_N^d} |f(x)|^p \right)^{\frac{1}{p}},$$

Proof of Uncertainty Principle via Restriction I (continued)

- and by the support assumption, this quantity is equal to

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$$|f(x)| \leq |S| \cdot C_{p,q} \cdot N^{-d} \cdot \left(\sum_{x \in E} |f(x)|^p \right)^{\frac{1}{p}}.$$

- Raising both sides to the power of p, summing over E, and dividing both sides of the resulting inequality by $\sum_{x \in E} |f(x)|^p$, we obtain

$$|S|^p \cdot |E| \cdot C_{p,q}^p \geq N^{dp}.$$

Proof of Uncertainty Principle via Restriction I (finale)

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as desired.

- This completes the proof of the Uncertainty Principle via Restriction Theory.

Proof of the universal restriction theorem

- We have

$$\sum_{m \in S} |\widehat{f}(m)|^2 = \sum_m 1_S(m)\widehat{f}(m)g(m),$$

where

$$g(m) = \overline{1_S \widehat{f}(m)}.$$

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- The expression above equals
$$\sum_x f(x)\widehat{1_S g}(x) \leq ||f||_{L^{\frac{4}{3}}(\mathbb{Z}_N^d)} \cdot \left(\sum_{x\in\mathbb{Z}_N^d} |\widehat{1_S g}(x)|^4\right)^{\frac{1}{4}}.$$

Proof of the universal restriction theorem (continued)

- We have

$$\sum_{x \in \mathbb{Z}_N^d} \left|\widehat{1_S g}(x)\right|^4$$

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- The quantity above is bounded by

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- It follows that

$$\left(\sum_{x \in \mathbb{Z}_N^d} |\widehat{1_S g}(x)|^4 \right)^{\frac{1}{4}} \leq N^{-\frac{d}{4}} \cdot \left(\max_{U \subset S} \frac{\Lambda(U)}{|U|^2} \right)^{\frac{1}{4}} \cdot ||g||_{L^2(\mathbb{Z}_N^d)}.$$

Proof of the universal restriction theorem (continued)

- Putting everything together, we see that

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- $$= \left(\frac{|S|}{N^{\frac{d}{2}}}\right)^{-\frac{1}{2}} \cdot \left(\max_{U\subset S}\frac{\Lambda(U)}{|U|^2}\right)^{\frac{1}{4}} \cdot N^{-\frac{d}{2}} \cdot \left(\sum_{x\in\mathbb{Z}_N^d}|f(x)|^{\frac{4}{3}}\right)^{\frac{3}{4}}.$$

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$$\left(\sum_{m\in S}|\widehat{f}(m)|^2\right)^{\frac{1}{2}}$$
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Proof of the additive energy uncertainty principle (continued)

- Since $\widehat{f}$ is supported in S, we can apply Plancherel and obtain

$$\left(\sum_{x\in E}|f(x)|^2\right)^{\frac{1}{2}}$$

$$\leq |S|^{\frac{1}{2}}\cdot\left(\frac{|S|}{N^{\frac{d}{2}}}\right)^{-\frac{1}{2}}\cdot\left(\max_{U\subset\Sigma}\frac{\Lambda(U)}{|U|^2}\right)^{\frac{1}{4}}\cdot N^{-\frac{d}{2}}\cdot\left(\sum_{x\in E}|f(x)|^{\frac{4}{3}}\right)^{\frac{3}{4}}.$$

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- Applying Hölder's inequality, we obtain

$$\left(\sum_{x\in E} |f(x)|^2\right)^{\frac{1}{2}}$$

$$\leq |S|^{\frac{1}{2}} \cdot \left(\frac{|S|}{N^{\frac{d}{2}}}\right)^{-\frac{1}{2}} \cdot \left(\max_{U\subset S} \frac{\Lambda(U)}{|U|^2}\right)^{\frac{1}{4}} \cdot |E|^{\frac{1}{4}} \cdot N^{-\frac{d}{2}} \cdot \left(\sum_{x\in E} |f(x)|^2\right)^{\frac{1}{2}}.$$

Proof of the additive energy uncertainty principle (continued)

- It follows that

$$N^{\frac{d}{4}} \leq \left(\max_{U \subset S} \frac{\Lambda(U)}{|U|^2} \right)^{\frac{1}{4}} \cdot |E|^{\frac{1}{4}},$$

  and we conclude that

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- Exactly the same argument with f replaced by $\widehat{f}$ and S replaced by E yields

$$N^d \leq |S| \cdot \max_{F \subset E} \frac{\Lambda(F)}{|F|^2}.$$

Another version of the additive energy uncertainty principle

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Theorem

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$$N^d \le \Lambda^{\frac{\alpha}{3}}(E)\Lambda^{\frac{1-\alpha}{3}}(S)|E|^{1-\alpha}|S|^{\alpha}.$$

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Proof of the alternate version of the additive energy uncertainty principle

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Proof of the alternate version of the additive energy uncertainty principle (continued)

- We have

$$\sum_{m \in S} |\widehat{f}(m)|^4$$

$$= N^{-2d} \sum_{m \in \mathbb{Z}_N^d} \sum_{x,y,x',y' \in E} \chi((x+y-x'-y') \cdot m)\overline{f(x)f(y)}f(x')f(y')$$

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- $$= N^{-d} \sum_{x+y=x'+y'; x,y,x',y' \in E} \overline{f(x)f(y)} f(x') f(y')$$

- $$\leq N^{-d} \cdot \Lambda(E) \cdot ||f||^4_{L^\infty(E)}.$$

Proof of the alternate version of the additive energy uncertainty principle (continued)

- Putting everything together, we see that

$$|f(x)| \leq N^{-\frac{d}{2}} \cdot |S|^{\frac{3}{4}} \cdot N^{-\frac{d}{4}} \cdot \Lambda^{\frac{1}{4}}(E) \cdot ||f||_{L^\infty(E)}.$$

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- Taking the maximum over $x \in E$ and cancelling the $L^\infty(E)$ norms, we obtain

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- Reversing the roles of E and S, we obtain

$$N^d \leq \Lambda^{\frac{1}{3}}(S) \cdot |E|, \text{ which completes the proof.}$$

Bourgain's Λ_q theorem - general formulation

- Jean Bourgain proved that if G is a locally compact abelian group, $\phi_1, \dots, \phi_n$ are orthogonal functions with $||\phi_j||_\infty \leq 1$, the for a generic set $S \subset \{1, 2, \dots, n\}$ of size $\approx n^{\frac{2}{q}}$, $q > 2$,

$$\left|\left|\sum_{i \in S} a_i \phi_i\right|\right|_{L^q(G)} \leq C(q) \cdot \left(\sum_{i \in S} |a_i|^2\right)^{\frac{1}{2}},$$

where $C(q)$ depends only on q.

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  where $C(q)$ depends only on q.

- As we shall see, this result has a beautiful built-in uncertainty principle.

Bourgain's Λ_q theorem

- It is a consequence of Bourgain's celebrated Λ_p theorem in locally compact abelian groups that if $f : \mathbb{Z}_N^d \to \mathbb{C}$ and $\widehat{f}$ **is supported in** S, then for a "generic" set of size $\approx N^{\frac{2d}{q}}$, $2 < q < \infty$,

$$\left(\frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} |f(x)|^q \right)^{\frac{1}{q}} \leq K_q(S) \left(\frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} |f(x)|^2 \right)^{\frac{1}{2}},$$

with $K_q(S)$ independent of N.

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with $K_q(S)$ independent of N.

- It is not difficult to see that this inequality implies that the support of f must be a positive proportion of $\mathbb{Z}_N^d$.

A direct consequence of Bourgain's Λ_q theorem

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A direct consequence of Bourgain's Λ_q theorem

- Suppose that S is generic, as in Bourgain's theorem.
- Suppose that f is supported in $E \subset \mathbb{Z}_N^d$ and $\widehat{f}$ is supported in S. Bourgain's theorem implies that
- $$N^{-\frac{d}{q}} \cdot |E|^{\frac{1}{q}} \left(\frac{1}{|E|} \sum_{x \in E} |f(x)|^q \right)^{\frac{1}{q}}$$
$$\leq K_q(S) N^{-\frac{d}{2}} \cdot |E|^{\frac{1}{2}} \left(\frac{1}{|E|} \sum_{x \in E} |f(x)|^2 \right)^{\frac{1}{2}}.$$

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- It follows that

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- It follows that if $\widehat{f}$ is supported in a generic set of size $\approx N^{d-\epsilon}$, then f is supported on a positive proportion of $\mathbb{Z}_N^d$.

- We conclude that if we send the Fourier transform of a signal f supported on a set of size $o(N^d)$, and the frequencies in $S \subset \mathbb{Z}_N^d$ satisfying a Λ_q, $q > 2$, inequality are missing, we can recover f exactly and uniquely with very high probability.

Annihilating pairs

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- Let $E, S \subset \mathbb{R}$ have finite measure. Then there exists a constants $c > 0$ such that

$$||f||_{L^2(\mathbb{R})} \leq e^{c|E||S|} \left(||f||_{L^2(E^c)} + ||\widehat{f}||_{L^2(S^c)} \right).$$

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- We may discuss the continuous case in more detail later in these lectures.

- For the moment we immerse ourselves back in the world of finite signals.

Annihilating pairs: Ghobber and Jaming

- Let $f : \mathbb{Z}_N^d \to \mathbb{C}$. Ghobber and Jaming proved in 2011 that if $E, S \subset \mathbb{Z}_N^d$, $|E| \cdot |S| < N^d$, then

$$||f||_{L^2(\mathbb{Z}_N^d)} \leq \left(1 + \frac{1}{1 - \sqrt{\frac{|E||S|}{N^d}}}\right) \cdot \left(||f||_{L^2(E^c)} + ||\widehat{f}||_{L^2(S^c)}\right).$$

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- Observe that this result easily implies the classical uncertainty principle since if f is supported in E, $\widehat{f}$ is supported in S, and

$$|E| \cdot |S| < N^d,$$

then the right hand side of the inequality above is 0. Hence the left hand side is also 0 and the uncertainty principle is established.

Proof of the Ghobber-Jaming result

- We have

$$||\widehat{1_E f}||_{L^2(S)} \leq N^{-\frac{d}{2}} \cdot |S|^{\frac{1}{2}} \cdot ||f||_{L^1(E)}$$
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-

$$\geq ||f||_{L^2(E)} \left(1 - N^{-\frac{d}{2}} \cdot |S|^{\frac{1}{2}} \cdot |E|^{\frac{1}{2}}\right).$$

Proof of the Ghobber-Jaming result (continued)

- We are almost ready to drive for the finish line. By the triangle inequality,

$$||f||_{L^2(\mathbb{Z}_N^d)} \leq ||f||_{L^2(E)} + ||f||_{L^2(E^c)}$$

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Proof of the Ghobber-Jaming result (continued)

- $$\leq \left(||\widehat{f}||_{L^2(S^c)} + ||f||_{L^2(E^c)} \right) \cdot \frac{1}{1 - \sqrt{\frac{|E||S|}{N^d}}} + ||f||_{L^2(E^c)}$$

- $$\left(1 + \frac{1}{1 - \sqrt{\frac{|E||S|}{N^d}}} \right) \cdot \left(||f||_{L^2(E^c)} + ||\widehat{f}||_{L^2(S^c)} \right),$$

  and the proof is complete.

Annihilating pairs and structure of sets

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- Just as we were able prove a stronger uncertainty principle in the presence of limited additive structure, we can do the same in the case of annihilating pairs inequalities.

- The following is a recent result due to A.I., P. Jaming and A. Mayeli. Suppose that a (p,q) Fourier restriction estimate holds for $S \subset \mathbb{Z}_N^d$, $1 \le p \le 2 \le q$, with norm $C_{p,q}$. Then

$$||f||_{L^2(\mathbb{Z}_N^d)} \le \left(1 + \frac{1}{1 - \sqrt{\frac{C_{p,q}^2 |E|^{\frac{2-p}{p}} |S|}{N^d}}} \right) \cdot \left(||f||_{L^2(E^c)} + ||\widehat{f}||_{L^2(S^c)} \right),$$

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- provided that

$$|E|^{\frac{2-p}{p}} |S| < \frac{N^d}{C_{p,q}^2}.$$

The case $1 \leq p \leq q \leq 2$

- If $1 \leq p \leq q \leq 2$ and if a (p, q) Fourier restriction estimate holds for S,

$$||f||_{L^2(\mathbb{Z}_N^d)} \leq \left(1 + \frac{|E|^{\frac{1}{2}-\frac{1}{q'}}}{1 - \left(\frac{|S||E|^{\frac{(q'-p)q}{q'p}} C_{p,q}^q}{N^d} \right)^{\frac{1}{q}}} \right) \left(||f||_{L^2(E^c)} + ||\widehat{f}||_{L^2(S^c)} \right),$$

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- provided that

$$|E|^{\frac{(q'-p)q}{q'p}} \cdot |S| < \frac{N^d}{C_{p,q}^q}.$$

Proof of the A.I.-Jaming-Mayeli result

- We first handle the case $1 \le p \le 2 \le q$. By the restriction assumption,

$$||\widehat{1_E f}||_{L^2(S)} = |S|^{\frac{1}{2}} ||\widehat{1_E f}||_{L^2(\mu_S)} \le |S|^{\frac{1}{2}} ||\widehat{1_E f}||_{L^q(\mu_S)}$$

$$\le |S|^{\frac{1}{2}} \cdot C_{p,q} N^{-\frac{d}{2}} ||f||_{L^p(E)}$$

by assumption.

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$$\leq |S|^{\frac{1}{2}} \cdot C_{p,q} N^{-\frac{d}{2}} ||f||_{L^p(E)}$$

by assumption.

- By Holder's inequality, this quantity is bounded by

$$C_{p,q}|S|^{\frac{1}{2}} N^{-\frac{d}{2}} |E|^{\frac{2-p}{2p}} ||f||_{L^2(E)} = \sqrt{\frac{C_{p,q}^2 |S||E|^{\frac{2-p}{p}}}{N^d}} ||f||_{L^2(E)}.$$

Proof of the A.I.-Jaming-Mayeli result (continued)

- On the other hand,

$$||\widehat{1_E f}||_{L^2(S^c)} \geq ||\widehat{1_E f}||_{L^2(\mathbb{Z}_N^d)} - ||\widehat{1_E f}||_{L^2(S)}$$

$$\geq \left(1 - \sqrt{\frac{C_{p,q}^2 |S||E|^{\frac{2-p}{p}}}{N^d}}\right) ||f||_{L^2(E)}.$$

We are now ready for the conclusion of the proof. We have

$$||f||_{L^2(\mathbb{Z}_N^d)} \leq ||f||_{L^2(E)} + ||f||_{L^2(E^c)}$$

$$\leq \left(1 - \sqrt{\frac{C_{p,q}^2 |S||E|^{\frac{2-p}{p}}}{N^d}}\right)^{-1} ||\widehat{1_E f}||_{L^2(S^c)} + ||f||_{L^2(E^c)}.$$

Proof of the A.I.-Jaming-Mayeli result (continued)

- We are left to unravel the quantity $||\widehat{1_E f}||_{L^2(S^c)}$. We have

$$||\widehat{1_E f}||_{L^2(S^c)} = ||1_{S^c}\widehat{f} - 1_{S^c}\widehat{1_{E^c} f}||_{L^2(\mathbb{Z}_N^d)}$$

$$\leq ||\widehat{f}||_{L^2(S^c)} + ||f||_{L^2(E^c)}.$$

Plugging this back into above, we have

$$||f||_{L^2(\mathbb{Z}_N^d)} \leq$$

$$\leq \left(1 - \sqrt{\frac{C_{p,q}^2 |S||E|^{\frac{2-p}{p}}}{N^d}}\right)^{-1} \left(||\widehat{f}||_{L^2(S^c)} + ||f||_{L^2(E^c)}\right) + ||f||_{L^2(E^c)}$$

and the case $1 \leq p \leq 2 \leq q$ is established.

Proof of the A.I.-Jaming-Mayeli result (continued)

- We now handle the case $1 \leq p \leq q \leq 2$. By assumption, we have

$$||\widehat{1_E f}||_{L^q(S)} \leq |S|^{\frac{1}{q}} C_{p,q} N^{-\frac{d}{2}} ||f||_{L^p(E)}$$

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- $$\leq |S|^{\frac{1}{q}} |E|^{\frac{1}{p}-\frac{1}{2}} C_{p,q} N^{-\frac{d}{2}} ||f||_{L^2(E)}.$$

Lemma (Hausdorff-Young inequality)

Suppose that $f : \mathbb{Z}_N^d \to \mathbb{C}$ *and* $1 \leq p \leq 2$. *Then*

$$||\widehat{f}||_{L^{p'}(\mathbb{Z}_N^d)} \leq N^{-\frac{d}{2}\left(\frac{2-p}{p}\right)} ||f||_{L^p(\mathbb{Z}_N^d)}.$$

-

Proof of the A.I.-Jaming-Mayeli result (continued)

- The case $p = 1$ follows by the triangle inequality and the definition of the Fourier transform. The case $p = 2$ is Plancherel. The result follows by Riesz-Thorin interpolation theorem.

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- The case $p = 1$ follows by the triangle inequality and the definition of the Fourier transform. The case $p = 2$ is Plancherel. The result follows by Riesz-Thorin interpolation theorem.

- Using Hausdorff-Young, we have

$$||\widehat{1_E f}||_{L^q(\mathbb{Z}_N^d)} \geq N^{\frac{d}{2}\left(\frac{2-q}{q}\right)} ||f||_{L^{q'}(E)}$$

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- $$\geq N^{\frac{d}{2}\left(\frac{2-q}{q}\right)} |E|^{\frac{1}{2}-\frac{1}{q'}} ||f||_{L^2(E)}.$$

Proof of the A.I.-Jaming-Mayeli result (continued)

- Combining, we obtain

$$||f||_{L^2(E)} \leq \frac{||\widehat{1_E f}||_{L^q(S^c)}}{N^{\frac{d}{2}\left(\frac{2-q}{q}\right)}|E|^{\frac{1}{2}-\frac{1}{q'}} - |S|^{\frac{1}{q}}|E|^{\frac{1}{p}-\frac{1}{2}} C_{p,q} N^{-\frac{d}{2}}}.$$

Proof of the A.I.-Jaming-Mayeli result (continued)

- Combining, we obtain

$$||f||_{L^2(E)} \leq \frac{||\widehat{1_E f}||_{L^q(S^c)}}{N^{\frac{d}{2}\left(\frac{2-q}{q}\right)}|E|^{\frac{1}{2}-\frac{1}{q'}} - |S|^{\frac{1}{q}}|E|^{\frac{1}{p}-\frac{1}{2}} C_{p,q} N^{-\frac{d}{2}}}.$$

- We now unravel $||\widehat{1_E f}||_{L^q(S^c)}$. We have

$$||\widehat{1_E f}||_{L^q(S^c)} = ||\widehat{f} - \widehat{1_{E^c} f}||_{L^q(S^c)}$$

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- Combining, we obtain

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-

$$\leq ||\widehat{f}||_{L^q(S^c)} + ||\widehat{1_{E^c} f}||_{L^q(S^c)}$$

Proof of the A.I.-Jaming-Mayeli result (continued)

- $$\leq |S^c|^{\frac{1}{q}-\frac{1}{2}} \left(||\widehat{f}||_{L^2(S^c)} + ||f||_{L^2(E^c)} \right).$$

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- Rearranging the terms yields the conclusion of the case $1 \leq p \leq q \leq 2$.

The additive energy annihilation inequality

Theorem

(A.I., P. Jaming, and A. Mayeli (2024)) Let $f : \mathbb{Z}_N^d \to \mathbb{C}$. *Let* $E, S \subset \mathbb{Z}_N^d$ *such that*

$$\max_{U \subset S} \frac{\Lambda(U)}{|U|^2} \cdot |E| < N^d.$$

Then

$$||f||_{L^2(\mathbb{Z}_N^d)} \leq C_{ann} \left(||f||_{L^2(E^c)} + ||\widehat{f}||_{L^2(S^c)} \right),$$

where C_{ann} *may be taken to be*

$$1 + \frac{1}{1 - \sqrt{\frac{\left(\max_{U \subset S} \frac{\Lambda(U)}{|U|^2}\right)^{\frac{1}{2}} |E|^{\frac{1}{2}}}{N^{\frac{d}{2}}}}}.$$

Proof of the additive energy annihilation inequality

- This result follows by inserting

$$\left(\frac{|S|}{N^{\frac{d}{2}}}\right)^{-\frac{1}{2}} \cdot \left(\max_{U \subset S} \frac{\Lambda(U)}{|U|^2}\right)^{\frac{1}{4}}$$

from the universal restriction theorem in place of the constant $C_{\frac{4}{3},2}$ in the restriction annihilation inequality above.

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- Similar results can be obtained in Euclidean space as well, and we shall talk about that if time allows.

A symmetrized extension

- We can symmetrize, as before, and replace C_{ann} above with

$$\left(1+\frac{1}{1-\sqrt{\frac{\left(\max_{U\subset S}\frac{\Lambda(U)}{|U|^2}\right)^{\frac{1}{2}}|E|^{\frac{1}{2}}}{N^{\frac{d}{2}}}}}\right)^{\alpha}$$

times

$$\left(1+\frac{1}{1-\sqrt{\frac{\left(\max_{F\subset E}\frac{\Lambda(F)}{|F|^2}\right)^{\frac{1}{2}}|S|^{\frac{1}{2}}}{N^{\frac{d}{2}}}}}\right)^{1-\alpha}$$

for any $\alpha \in [0,1]$ provided that

A symmetrized extension (continued)

- $$\max_{U \subset S} \frac{\Lambda(U)}{|U|^2} \cdot |E| < N^d$$

  and

A symmetrized extension (continued)

- $$\max_{U \subset S} \frac{\Lambda(U)}{|U|^2} \cdot |E| < N^d$$

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A symmetrized extension (continued)

- $$\max_{U \subset S} \frac{\Lambda(U)}{|U|^2} \cdot |E| < N^d$$

  and

- $$\max_{F \subset E} \frac{\Lambda(F)}{|F|^2} \cdot |S| < N^d.$$

- As usual, the corresponding uncertainty principle can be deduced by assuming that f is supported in E and $\widehat{f}$ is supported in S.

An L^p annihilating pairs inequality

Theorem

(A.I., P. Jaming and. A. Mayeli (2024)) Let $f : \mathbb{Z}_N^d \to \mathbb{C}$. Let $E, S \subset \mathbb{Z}_N^d$ such that S satisfies the (p, q) restriction estimate for some $1 \leq p \leq 2 \leq q$, and $|E|^{2-p} \cdot |S| < \frac{N^d}{C_{p,q}^p}$. Then for $1 \leq p \leq 2$, $||f||_{L^{p'}(\mathbb{Z}_N^d)}$ is bouned by

$$\frac{N^{-d\left(\frac{1}{2}-\frac{1}{p'}\right)}}{1-\left(\frac{|E|^{2-p}|S|C_{p,q}^p}{N^d}\right)^{\frac{1}{p}}}||\widehat{f}||_{L^p(S^c)} + \left(1+\frac{1}{1-\left(\frac{|E|^{2-p}|S|C_{p,q}^p}{N^d}\right)^{\frac{1}{p}}}\right)||f||_{L^{p'}(E^c)}.$$

Since $(1, q)$ restriction estimate always holds with $C_{1,q} = 1$, then for any sets $E, S \subset \mathbb{Z}_N^d$ such that $|E||S| < N^d$, $||f||_{L^\infty(\mathbb{Z}_N^d)}$ is bounded by

$$\frac{N^{-\frac{d}{2}}}{1-\frac{|E||S|}{N^d}}||\widehat{f}||_{L^1(S^c)} + \left(1+\frac{1}{1-\frac{|E||S|}{N^d}}\right)||f||_{L^\infty(E^c)}.$$

Proof of the L^p annihilating pairs inequality

- By the (p, q) restriction bound, we have

$$||\widehat{1_E f}||_{L^p(S)} \leq |S|^{\frac{1}{p}} \cdot \left(\frac{1}{|S|} \sum_{m \in S} |\widehat{1_E f}(m)|^q \right)^{\frac{1}{q}}$$

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- $$\leq C_{p,q} N^{-\frac{d}{2}} |S|^{\frac{1}{p}} ||f||_{L^p(E)}$$

- $$\leq C_{p,q} N^{-\frac{d}{2}} |S|^{\frac{1}{p}} |E|^{\frac{2-p}{p}} ||f||_{L^{p'}(E)}.$$

Proof of the L^p annihilating pairs inequality (continued)

- On the other hand,

$$||\widehat{1_E f}||_{L^p(S^c)} \geq ||\widehat{1_E f}||_{L^p(\mathbb{Z}_N^d)} - ||\widehat{1_E f}||_{L^p(S)}$$

Proof of the L^p annihilating pairs inequality (continued)

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- $$\geq N^{\frac{d}{2}\left(1-\frac{2}{p'}\right)}||f||_{L^{p'}(E)} - C_{p,q}N^{-\frac{d}{2}}|S|^{\frac{1}{p}}|E|^{\frac{2-p}{p}}||f||_{L^{p'}(E)}$$

Proof of the L^p annihilating pairs inequality (continued)

- On the other hand,

$$||\widehat{1_E f}||_{L^p(S^c)} \geq ||\widehat{1_E f}||_{L^p(\mathbb{Z}_N^d)} - ||\widehat{1_E f}||_{L^p(S)}$$

- $$\geq N^{\frac{d}{2}\left(1-\frac{2}{p'}\right)} ||f||_{L^{p'}(E)} - C_{p,q} N^{-\frac{d}{2}} |S|^{\frac{1}{p}} |E|^{\frac{2-p}{p}} ||f||_{L^{p'}(E)}$$

- $$= N^{\frac{d}{2}\left(1-\frac{2}{p'}\right)} \left(1 - \left(\frac{|E|^{2-p}|S|C_{p,q}^p}{N^d}\right)^{\frac{1}{p}}\right) ||f||_{L^{p'}(E^c)},$$

where in the second line we used the Hausdorff-Young inequality.

Proof of the L^p annihilating pairs inequality (continued)

- Observe that

$$||\widehat{1_E f}||_{L^p(S^c)} = ||\widehat{f} - \widehat{1_{E^c} f}||_{L^p(S^c)} \leq ||\widehat{f}||_{L^p(S^c)} + ||\widehat{1_{E^c} f}||_{L^p(S^c)}$$

Proof of the L^p annihilating pairs inequality (continued)

- Observe that

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-

$$\leq ||\widehat{f}||_{L^p(S^c)} + \left(\sum_{m \in S^c} |\widehat{1_{E^c} f}(m)|^p \right)^{\frac{1}{p}}$$

$$\leq ||\widehat{f}||_{L^p(S^c)} + |S^c|^{\frac{1}{p}} \left(\frac{1}{|S^c|} \sum_{m \in S^c} |\widehat{1_{E^c} f}(m)|^p \right)^{\frac{1}{p}}$$

Proof of the L^p annihilating pairs inequality (continued)

- $$\leq ||\widehat{f}||_{L^p(S^c)} + |S^c|^{\frac{1}{p}} \left(\frac{1}{|S^c|} \sum_{m \in S^c} |\widehat{1_{E^c} f}(m)|^{p'} \right)^{\frac{1}{p'}}$$
  $$= ||\widehat{f}||_{L^p(S^c)} + |S^c|^{\frac{1}{p} - \frac{1}{p'}} \left(\sum_{m \in S^c} |\widehat{1_{E^c} f}(m)|^{p'} \right)^{\frac{1}{p'}}$$

Proof of the L^p annihilating pairs inequality (continued)

- $$\leq ||\widehat{f}||_{L^p(S^c)} + |S^c|^{\frac{1}{p}} \left(\frac{1}{|S^c|} \sum_{m \in S^c} |\widehat{1_{E^c} f}(m)|^{p'} \right)^{\frac{1}{p'}}$$
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- $$\leq ||\widehat{f}||_{L^p(S^c)} + |S^c|^{\frac{1}{p} - \frac{1}{p'}} \cdot N^{-\frac{d}{2}\left(1 - \frac{2}{p'}\right)} ||f||_{L^p(E^c)}$$

Proof of the L^p annihilating pairs inequality (continued)

- $$\leq ||\widehat{f}||_{L^p(S^c)} + |S^c|^{\frac{1}{p}} \left(\frac{1}{|S^c|} \sum_{m \in S^c} |\widehat{1_{E^c} f}(m)|^{p'} \right)^{\frac{1}{p'}}$$
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- $$\leq ||\widehat{f}||_{L^p(S^c)} + |S^c|^{\frac{1}{p} - \frac{1}{p'}} \cdot N^{-\frac{d}{2}\left(1 - \frac{2}{p'}\right)} ||f||_{L^p(E^c)}$$
- $$\leq ||\widehat{f}||_{L^p(S^c)} + N^{d\left(\frac{1}{p} - \frac{1}{2}\right)} ||f||_{L^p(E^c)}.$$

Proof of the L^p annihilating pairs inequality (continued)

- $$\leq ||\widehat{f}||_{L^p(S^c)} + |S^c|^{\frac{1}{p}} \left(\frac{1}{|S^c|} \sum_{m \in S^c} |\widehat{1_{E^c} f}(m)|^{p'} \right)^{\frac{1}{p'}}$$

  $$= ||\widehat{f}||_{L^p(S^c)} + |S^c|^{\frac{1}{p} - \frac{1}{p'}} \left(\sum_{m \in S^c} |\widehat{1_{E^c} f}(m)|^{p'} \right)^{\frac{1}{p'}}$$

- $$\leq ||\widehat{f}||_{L^p(S^c)} + |S^c|^{\frac{1}{p} - \frac{1}{p'}} \cdot N^{-\frac{d}{2}\left(1 - \frac{2}{p'}\right)} ||f||_{L^p(E^c)}$$

- $$\leq ||\widehat{f}||_{L^p(S^c)} + N^{d\left(\frac{1}{p} - \frac{1}{2}\right)} ||f||_{L^p(E^c)}.$$

- By the triangle inequality,

  $$||f||_{L^{p'}(\mathbb{Z}_N^d)} \leq ||f||_{L^{p'}(E)} + ||f||_{L^{p'}(E^c)}$$

Proof of the L^p annihilating pairs inequality (continued)

- $$\leq \frac{N^{-d\left(\frac{1}{2}-\frac{1}{p'}\right)}}{1-\left(\frac{|E|^{2-p}|S|C_{p,q}^{p}}{N^d}\right)^{\frac{1}{p}}} ||\widehat{1_E f}||_{L^p(S^c)} + ||f||_{L^{p'}(E^c)}$$

Proof of the L^p annihilating pairs inequality (continued)

- $$\leq \frac{N^{-d\left(\frac{1}{2}-\frac{1}{p'}\right)}}{1-\left(\frac{|E|^{2-p}|S|C_{p,q}^p}{N^d}\right)^{\frac{1}{p}}}||\widehat{1_E f}||_{L^p(S^c)} + ||f||_{L^{p'}(E^c)}$$

- $$\leq \frac{N^{-d\left(\frac{1}{2}-\frac{1}{p'}\right)}}{1-\left(\frac{|E|^{2-p}|S|C_{p,q}^p}{N^d}\right)^{\frac{1}{p}}} \cdot \left(||\widehat{f}||_{L^p(S^c)} + N^{d\left(\frac{1}{p}-\frac{1}{2}\right)}||f||_{L^p(E^c)}\right) + ||f||_{L^{p'}(E^c)}$$

Proof of the L^p annihilating pairs inequality (continued)

- $$\leq \frac{N^{-d\left(\frac{1}{2}-\frac{1}{p'}\right)}}{1-\left(\frac{|E|^{2-p}|S|C^p_{p,q}}{N^d}\right)^{\frac{1}{p}}}||\widehat{f}||_{L^p(S^c)}$$

$$+\left(1+\frac{1}{1-\left(\frac{|E|^{2-p}|S|C^p_{p,q}}{N^d}\right)^{\frac{1}{p}}}\right)||f||_{L^{p'}(E^c)},$$

and the proof is complete.

A consequence of annihilating pairs inequalities

- The following result was originally proven directly by A.I. and A. Mayeli earlier this year, but it also follows directly from the annihilating pairs inequalities we just proved.

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Theorem

Suppose that $f : \mathbb{Z}_N^d \to \mathbb{C}$ is supported in $E \subset \mathbb{Z}_N^d$, and $\hat{f} : \mathbb{Z}_N^d \to \mathbb{C}$ is supported in $S \subset \mathbb{Z}_N^d$. Suppose S satisfies the (p,q) restriction estimate with norm $C_{p,q}$, $1 \leq p \leq q$, $p \leq 2$.

i) If $q \geq 2$, then

$$|E|^{\frac{2-p}{p}} \cdot |S| \geq \frac{N^d}{C_{p,q}^2}.$$

ii) If $1 \leq p \leq q \leq 2$, then

$$|E|^{\frac{(q'-p)q}{q'p}} \cdot |S| \geq \frac{N^d}{C_{p,q}^q}.$$

From Restriction to Exact Recovery

Corollary

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ with support $supp(f) = E$. Let r be another signal with support of the same size such that $\widehat{r}(m) = \widehat{f}(m)$ for $m \notin S$, and 0 otherwise. Suppose $S \subset \mathbb{Z}_N^d$ satisfies the (p, q), $p < 2$, restriction estimate with uniform constant $C_{p,q}$. Then f can be reconstructed from r uniquely if

$$|E|^{\frac{1}{p}} \cdot |S| < \frac{N^d}{2^{\frac{1}{p}} C_{p,q}},$$

or if

$$|E|^{\frac{2-p}{p}} \cdot |S| < \frac{N^d}{2^{\frac{2-p}{p}} C_{p,q}^2} \text{ when } q \geq 2,$$

and

$$|E|^{\frac{(q'-p)q}{q'p}} \cdot |S| < \frac{N^d}{2^{\frac{(q'-p)q}{q'p}} C_{p,q}^q} \text{ when } q \leq 2.$$

Concentration inequality

- Donoho and Stark showed that if $f : \mathbb{Z}_N^d \to \mathbb{C}$, and $E, S \subset \mathbb{Z}_N^d$ such that f is concentrated in E at level ϵ_E in the sense that

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  with ϵ_E, ϵ_S both < 1, then
- $$\epsilon_E + \epsilon_S \geq 1 - \sqrt{\frac{|E||S|}{N^d}}.$$

Concentration inequality (continued)

- The following is a direct consequence of our annihilation pairs inequalities.

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Corollary

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ and suppose that f is L^2-concentrated on E at level $\epsilon_E > 0$ and $\widehat{f}$ is L^2-concentrated on S at level ϵ_S. Suppose that $S \subset \mathbb{Z}_N^d$ satisfying the (p, q) restriction estimate with norm $C_{p,q}$. Then

$$\epsilon_E + \epsilon_S \geq \frac{1}{1 + \frac{1}{1 - \sqrt{\frac{C_{p,q}^2 |E|^{\frac{2-p}{p}} |S|}{N^d}}}}.$$

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Concentration inequality (continued)

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-
- Note that in the case $p = 1$, when the restriction estimate always holds with constant $C_{1,q} = 1$, we recover a condition that is slightly stronger than the Donoho-Stark condition above.

Proof of the concentration inequality

- The concentration inequality and the assumptions on the concentration of f on E and concentration of $\widehat{f}$ on S imply that

$$||f||_{L^2(\mathbb{Z}_N^d)} \leq C_{ann}\left(||f||_{L^2(E^c)} + ||\widehat{f}||_{L^2(S^c)}\right)$$
$$\leq C_{ann}(\epsilon_E + \epsilon_S)||f||_{L^2(\mathbb{Z}_N^d)}.$$

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which implies that

- $$\epsilon_E + \epsilon_S \geq \frac{1}{C_{ann}},$$

and the proof is complete.

Arithmetic ideas and uncertinty

- In 2006, Terry Tao proved that if $f : \mathbb{Z}_p \to \mathbb{C}$, p prime, f is supported in E and $\widehat{f}$ is supported in S, then

$$|E| + |S| \geq p + 1.$$

Arithmetic ideas and uncertainty

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- The key element of the proof is a classical theorem due to Cebotarev which says that if $A, B \subset \mathbb{Z}_p$, $|A| = |B|$, then

$$det\{\chi(xm)\}_{x \in A, m \in B} \neq 0, \text{ where } \chi(t) = e^{\frac{2\pi it}{p}}.$$

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- Roy Meshulam used Tao's result and a beautiful iteration argument show that if $f : \mathbb{Z}_p^d \to \mathbb{C}$ is supported in E and $\widehat{f}$ is supported in S, then for $0 \leq j \leq d - 1$,

$$p^j|E| + p^{d-j-1}|S| \geq p^d + p^{d-1}.$$

Sketch of the proof of Cebotarev's theorem

- Observe that if $P(x_1, \dots, x_n)$ is a polynomial with integer entries, and

$$P(\omega_1, \dots, \omega_n) = 0,$$

where $\omega_1, \dots, \omega_n$ are roots of unity modulo p, then

$$P(1, \dots, 1) = 0.$$

Let $\omega_j = e^{\frac{2\pi i x_j}{p}}$. We must show that

$$det\{\omega_j^{\xi_k}\}_{1 \le j,k \le n} \neq 0.$$

Sketch of the proof of Cebotarev's theorem (continued)

- Define

$$D(z_1, \dots, z_n) = det\{z_j^{\xi_k}\}_{1 \le j,k \le n}$$

$$= P(z_1, \dots, z_n) \prod_{1 \le j < j' \le n} (z_j - z_{j'}).$$

The proof is completed by showing that $P(1, \dots, 1)$, which follows by a tedious calculation which reduces matters to the fact that the classical Vandermonde determinant $\neq 0$.

Proof of Tao's uncertainty principle

- Suppose not. We assume that $|E| \geq 1$ since otherwise there is nothing to prove. For every $m \notin S$, we have

$$0 = \widehat{f}(m) = p^{-\frac{1}{2}} \sum_{x \in E} \chi(-xm) f(x).$$

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- By Chebotarev's theorem, the resulting square matrix is invertible, which implies that $f(x) = 0$ for all $x \in E$. This completes the proof.

Two dimensions - the magic lemma

Lemma

(A.I., A. Mayeli, and J. Pakianathan (2017)) [Magic Lemma] Suppose that $f : \mathbb{Z}_p^2 \to \mathbb{Q}$*, p odd prime. Suppose that* $\widehat{f}(m) = 0$ *for some* $m \neq (0,0)$*. Then* $\widehat{f}(rm) = 0$ *for all* $r \neq 0$*.*

Moreover, if $f(x) = 1_E(x)$*, the indicator function of* $E \subset \mathbb{Z}_p^2$*, and* $\widehat{1}_E(m) = 0$ *for some* $m \neq (0,0)$*, then E is equidistributed on the p lines orthogonal to m.*

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Moreover, if $f(x) = 1_E(x)$, the indicator function of $E \subset \mathbb{Z}_p^2$, and $\widehat{1}_E(m) = 0$ for some $m \neq (0,0)$, then E is equidistributed on the p lines orthogonal to m.

-
- Suppose that $\widehat{1}_E(m) = 0$, as above, with $m \neq (0,0)$ and let $r \neq 0$. We have

$$\widehat{1}_E(rm) = p^{-2} \sum_t \zeta^{\frac{t}{r}} n(t/r) = p^{-2} \sum_t \zeta^t n(t) = 0.$$

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- Suppose that $\widehat{1}_E(m)=0$, as above, with $m \neq (0,0)$ and let $r \neq 0$. We have

$$\widehat{1}_E(rm)=p^{-2}\sum_t \zeta^{\frac{t}{r}} n(t/r) = p^{-2}\sum_t \zeta^t n(t) = 0.$$

- It follows that if $m \neq (0,0)$ is a zero of $\widehat{1}_E$, then so is every non-zero multiple of m.

Proof of the magic lemma

- Observe that

$$0 = \sum_t \zeta^t n(t) = n(0) + n(1)\zeta + n(2)\zeta^2 + \cdots + n(p-1)\zeta^{p-1}$$

says that ζ satisfies the polynomial of degree $p-1$ with coefficients given by $\{n(t)\}$.

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says that ζ satisfies the polynomial of degree $p-1$ with coefficients given by $\{n(t)\}$.

- The minimal polynomial of ζ is

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{p-1}.$$

Proof of the magic lemma

- Observe that

$$0 = \sum_t \zeta^t n(t) = n(0) + n(1)\zeta + n(2)\zeta^2 + \cdots + n(p-1)\zeta^{p-1}$$

says that ζ satisfies the polynomial of degree $p-1$ with coefficients given by $\{n(t)\}$.

- The minimal polynomial of ζ is

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{p-1}.$$

- We conclude that $n(t) = \textit{constant}$, so E has the **same** number of points on lines $\perp m$. In particular, $|E|$ is a multiple of p.

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- i) Spectral synthesis in $\mathbb{R}^d$ and connections with restriction theory.
- ii) The uncertainty principle on Riemannian manifolds.
- iii) A random variant of Shannon-Nyquist sampling on Riemannian manifolds and unique continuation of the Laplace-Beltrami operator.

Another version of the uncertainty principle

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- Suppose that $f \in L^1_{loc}(\mathbb{R}^d)$ and $\widehat{f}$ is supported in S is a k-dimensional submanifold of $\mathbb{R}^d$. Suppose further that $f \in L^p(\mathbb{R}^d)$ for some $p \leq \frac{2d}{k}$. Then $f \equiv 0$.

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- A natural question is whether the exponent $\frac{2d}{k}$ is **sharp**, and what does it have to with **restriction theory**? If $k = d-1$ and S^{d-1} is the unit sphere, $\frac{2d}{d-1}$ is the sharp conjectured exponent for the dual of the restriction conjecture.

Proof of the Agranovsky-Narayanan theorem

- Let $\chi \in C_0^\infty$, supported on the unit ball,

$$\int \chi(x)dx = 1,$$

$$\chi_\epsilon(x) = \epsilon^{-d}\chi(x/\epsilon).$$

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- By Plancherel,

$$||u_\epsilon||_2 = \left(\int |f(x)|^2 |\widehat{\chi}(\epsilon x)|^2 dx \right)^{\frac{1}{2}} \lesssim ||f||_p \cdot \epsilon^{-\frac{d}{p'}}.$$

Proof of the Agranovsky-Narayanan theorem (continued)

- Let ψ be a smooth cut-off function. We have

$$| < u_\epsilon, \psi > |^2 \leq ||u_\epsilon||_2^2 \cdot \int_{S^\epsilon} |\psi(\xi)|^2 d\xi,$$

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- The same argument works for any set of packing dimension k (not necessarily an integer).

Sharpness (or lack of it)

- If $S = S^{d-1}$, it is not difficult to see that the exponent $\frac{2d}{k} = \frac{2d}{d-1}$ is best possible since

$$\widehat{\sigma}_S(\xi) = J_{\frac{d-2}{2}}(|\xi|)|\xi|^{-\frac{d-2}{2}} \in L^p(\mathbb{R}^d) \text{ iff } p > \frac{2d}{d-1},$$

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- On the other hand, if

$$S = \left\{(t, t^2, \ldots, t^d) : t \in [0,1]\right\}, \; d \geq 3,$$

it is known that

$$\widehat{\sigma}_S \in L^p(\mathbb{R}^d) \text{ iff } p > \frac{d^2+d+2}{2} > \frac{2d}{k} = 2d.$$

A geometric approach to spectral synthesis

- Let $\widehat{f}$ be supported in S and let us cover S by a collection of **finitely overlapping** rectangles

$$\{R_{j,\delta}\}_{j=1}^{N(\delta)}, \ |R_{j,\delta}| \to 0 \text{ as } \delta \to 0.$$

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- Let $\mu_{j,\delta}$ denote a smooth partition of unity subordinate to $\{R_{j,\delta}\}_{j=1}^{N(\delta)}$. Since $\widehat{f}$ is supported in S, it is sufficient to consider

$$\widehat{f}(\xi) \cdot \sum_{j=1}^{N(\delta)} \mu_{j,\delta}(\xi), \text{ i.e.}$$

A geometric approach to spectral synthesis (continued)

- $$||f||_\infty \approx \left|\left| f * \sum_{j=1}^{N(\delta)} \widehat{\mu}_{j,\delta} \right|\right|_\infty \leq ||f||_p \cdot \left|\left| \sum_{j=1}^{N(\delta)} \widehat{\mu}_{j,\delta} \right|\right|_{p'} .$$

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- By Plancherel,

$$\left\| \sum_{j=1}^{N(\delta)} \widehat{\mu}_{j,\delta} \right\|_2 \approx \left(\sum_{j=1}^{N(\delta)} |R_{j,\delta}| \right)^{\frac{1}{2}} \equiv |S^{\delta}|^{\frac{1}{2}}.$$

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- Note that S^δ **is not necessarily** the δ-neighborhood of S.

A geometric approach to spectral synthesis (continued)

- On the other hand, since $R_{j,\delta}$'s are rectangles,

$$\left\| \sum_{j=1}^{N(\delta)} \widehat{\mu}_{j,\delta} \right\|_1 \lesssim \sum_{j=1}^{N(\delta)} |R_{j,\delta}| \cdot |R^*_{j,\delta}| = N(\delta).$$

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- The idea is to find the largest p for which this quantity $\to 0$ as $\delta \to 0$.

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- We conclude that
$$|S^\delta|^{\frac{1}{p}} \cdot (N(\delta))^{1-\frac{2}{p}} \approx \delta^{\frac{1}{p}},$$
which goes to 0 for any $p < \infty$.

A fun example

- Let $S = S^{d-1}$. Cover S by tangent $\delta^{\frac{1}{2}} \times \delta^{\frac{1}{2}} \times \dots \delta^{\frac{1}{2}} \times \delta$ finitely overlapping rectangles. It is not difficult to see that

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- It follows that the critical value for p is $\frac{2d}{d-1}$, which is consistent with Agranovsky-Narayanan's theorem.

An even more entertaining example

- Let $S = \{(t, t^2, \dots, t^d) : t \in [0,1]\}$. Cover S by $\delta^{\frac{1}{d}} \times \delta^{\frac{2}{d}} \times \cdots \times \delta$ tangent rectangles.

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$$p_{critical} = \frac{d^2+d+2}{2}.$$

Space curves

Theorem

(S. Guo, A. Iosevich, R. Zhang, and P. Zorich-Kranich (2023)) Let $d \geq 2$ be a positive integer and suppose that $1 \leq p < \frac{d^2+d+2}{2}$. If $f \in L^p(\mathbb{R}^d)$ and $\widehat{f}$ is supported on

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- Note that the Agranovsky-Narayanan theorem yields the same conclusion for $p < 2d$ in this case.
- We also note that $\frac{d^2+d+2}{2}$ is the optimal extension exponent (more on that in a moment).

Connections with the restriction conjecture

- On the very first page of these notes, we discussed the restriction conjecture, which says that if S^{d-1} is the unit sphere, then

$$\left(\int_{S^{d-1}} |\widehat{f}(\xi)|^r d\sigma_S(\xi)\right)^{\frac{1}{r}} \leq C_{p,r}\left(\int_{\mathbb{R}^d} |f(x)|^p dx\right)^{\frac{1}{p}}$$

whenever

$$p < \frac{2d}{d+1}, \; r \leq \frac{d-1}{d+1}p',$$

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- It is often convenient to state the dual of this inequality, the extension conjecture.

The extension conjecture

- The dual of the restriction conjecture above says that

$$||\widehat{f\sigma}||_{L^q(\mathbb{R}^d)} \leq C_{p,q}||f||_{L^p(S^{d-1})},$$

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- We call the inf of q's for which this estimate holds the critical extension exponent of S.

Extension versus spectral synthesis

- Based on examples we have so far, it seems reasonable to conjecture that if $\widehat{f}$ is supported in S, and $f \in L^p(\mathbb{R}^d)$ for p smaller than the critical extension exponent of S, then $f \equiv 0$.

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- I do not believe this conjecture. A potential counter-example is a compact strictly convex surface S, which has non-vanishing curvature in the sense that the volume of δ-caps is $\geq c\delta^{\frac{d+1}{2}}$ with $c > 0$ uniform.

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- I believe that it is possible to construct such a surface so that the critical extension exponent is $>> \frac{2d}{d-1}$.

Signal recovery on manifolds (joint work with A. Mayeli and E. Wyman)

- Let M be a compact Riemannian manifold without a boundary, and let $\{e_j\}_{j=1}^{\infty}$ be the family of L^2-normalized eigenfunctions of $\sqrt{-\triangle}$.

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- Suppose that A is a measurable subset of M and we wish to recover $1_A(x)$ from its Fourier coefficients, with frequencies in $\{j : \lambda_j \in S\}$ missing, where S is a subset of Λ, the set of eigenvalues of $\sqrt{-\triangle}$.

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- We have

$$1_A(x) = \sum_j < 1_A, e_j > e_j = \sum_{j \notin \{j:\lambda_j \in S\}} < 1_A, e_j > e_j +$$

$$+ \sum_{j \in \{j:\lambda_j \in S\}} < 1_A, e_j > e_j = I(x) + II(x).$$

Eigenvalues can be large

- We have

$$|II(x)| \leq |A|^{\frac{1}{2}} \cdot \left(\sum_{j \in \{j : \lambda_j \in S\}} |e_j(x)|^2 \right)^{\frac{1}{2}}.$$

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- If the manifold is **homogeneous** in the sense that there exists a **transitive group action** on M, the argument also goes through. But on general manifolds the situation is less clear.

Sampling on manifolds

- The basic question we ask is the following. Let (M, g) be a compact d-dimensional Riemannian manifold, as above, and let e_1, e_2, ..., e_n denote the eigenfunctions of the Laplace-Beltrami operator on M, where the corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ are not necessarily the *lowest* n eigenvalues.

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- When can we learn a function $f \in \text{span}\{e_1, \ldots, e_n\}$ by observing its value on some finite set of points $x_1, \ldots, x_m$?

Sampling on manifolds (continued)

- Note, given such an f, we need only identify its Fourier coefficients a_j in
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- But,

$$\begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_m) \end{bmatrix} = \begin{bmatrix} e_1(x_1) & \cdots & e_n(x_1) \\ e_1(x_2) & \cdots & e_n(x_2) \\ \vdots & & \vdots \\ e_1(x_m) & \cdots & e_n(x_m) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = A \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Hence, the recovery problem is equivalent to the matrix A having a left inverse. This necessitates $m \geq n$.

Sampling on manifolds (continued)

- The Nyquist-Shannon sampling theorem (ancient) says that if M is the one-dimensional torus and the frequencies of f are in $[-R, R]$, then we can recover f from any net of separation $\leq \frac{1}{2R}$.

Sampling on manifolds (continued)

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- This result was generalized to the setting of Riemannian manifolds by Pesenson (2008). In particular, if (M, g) is a d-dimensional Riemannian manifold and f is a finite linear combination of eigenfunctions $\{e_j\}$ with the corresponding eigenvalues bounded by R, then f can be recovered from $\approx R^d$ suitably separated samples.

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- This type of a result is quite efficient if the spectrum of the function consists of all the possible eigenfunctions with eigenvalues in a given range, but if the set of eigenvalues is relatively sparse, a much better result can expected. We will show that, if n points $x_1, \dots, x_n$ are selected randomly and independently with uniform probability from M, then A almost certainly has non-zero determinant.

Sampling on manifolds (continued)

Theorem

(A.I. and E. Wyman, 2024) Let (M, g) be a compact, connected Riemannian manifold without boundary, and $e_1, \ldots, e_n$ be an orthonormal set of Laplace-Beltrami eigenfunctions on M. If $x_1, \ldots, x_n$ are chosen independently and with uniform probability from M, then

$$\det \begin{bmatrix} e_1(x_1) & \cdots & e_n(x_1) \\ \vdots & & \vdots \\ e_1(x_n) & \cdots & e_n(x_n) \end{bmatrix} \neq 0$$

with probability 1.

Sampling on manifolds (continued)

Theorem

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$$\mathbb{P}\left\{ \left| \det \begin{bmatrix} e_1(x_1) & \cdots & e_n(x_1) \\ \vdots & & \vdots \\ e_1(x_n) & \cdots & e_n(x_n) \end{bmatrix} \right| \leq \epsilon \right\} \leq c\epsilon^{\frac{1}{k}},$$

where c is a universal constant.

-

Sampling on manifolds (continued)

Corollary

(A.I. and E. Wyman, 2024) Let (M, g) be a compact, connected Riemannian manifold without boundary, and $e_1, \dots, e_n$ be an orthonormal set of Laplace-Beltrami eigenfunctions on M. If $x_1, \dots, x_n$ are chosen independently and with uniform probability from M, then there exists a positive integer $k \geq 2$ such that

$$\mathbb{P}\left\{\lambda_{lowest}(x_1, \dots, x_n) \leq \epsilon\right\} \leq c\epsilon^{\frac{1}{nk}},$$

where c is a universal constant and $\lambda_{lowest}(x)$ is the smallest eigenvalue of the matrix

$$\begin{bmatrix} e_1(x_1) & \cdots & e_n(x_1) \\ \vdots & & \vdots \\ e_1(x_n) & \cdots & e_n(x_n) \end{bmatrix}.$$

Sampling on manifolds: key lemma

Lemma

If a finite linear combination of Laplace-Beltrami eigenfunctions vanishes to infinite order at a point in a connected, compact manifold, then it vanishes identically on the manifold.

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- The proof follows from the strong unique continuation property of solutions of the Laplace-Beltrami eigenfunction equations.


Eh...
thanks
a lot!