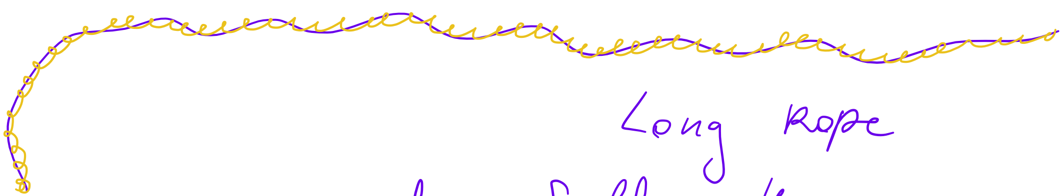
Summer school ICMU
functional isoperimetry, high-dimensional phenomena

I toy question (ancient Greece)



Long rope

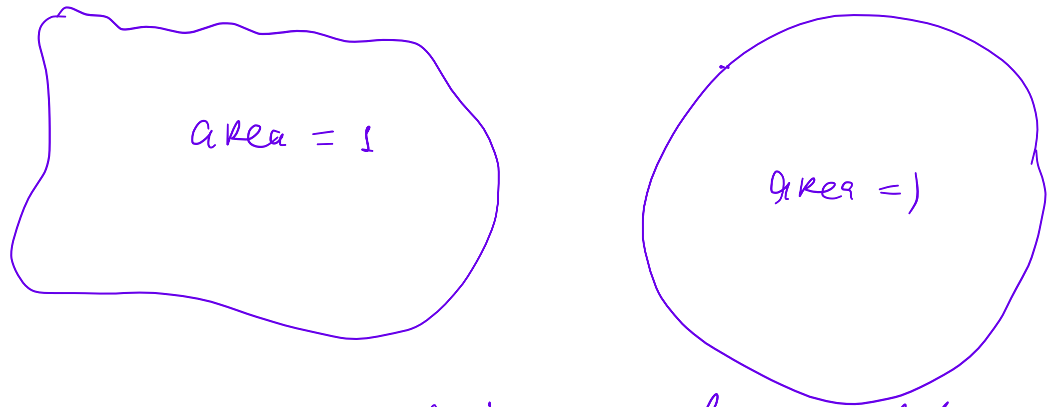
surround a field with it

The isoperimetric inequality

1837 Steiner
Rigorous proofs
in the 20th
century

Of all the shapes of fixed area

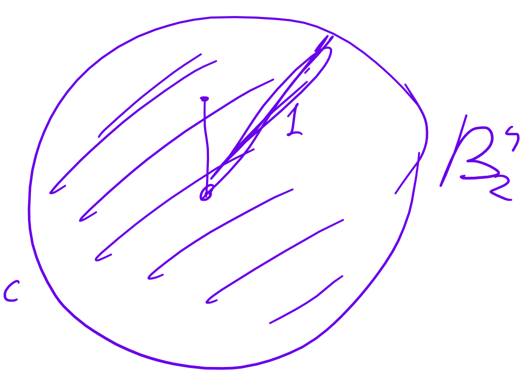
the disc (circle) has the smallest perimeter



More generally, $\mathbb{R}^n$ – n-dim Euclidean space

$B_2^n = \{x \in \mathbb{R}^n : |x| \le 1\}$, here unit ball

$|x| = \|x\|_2 = \sqrt{x_1^2 + \dots + x_n^2}$



Theorem (the isoperimetric inequality) informally

$K \subset \mathbb{R}^n$ is a measurable set

$Vol_n(K) = Vol_n(B_2^n)$

↑ n-dim volume

Then $perim(K) \ge perim(B_2^n)$

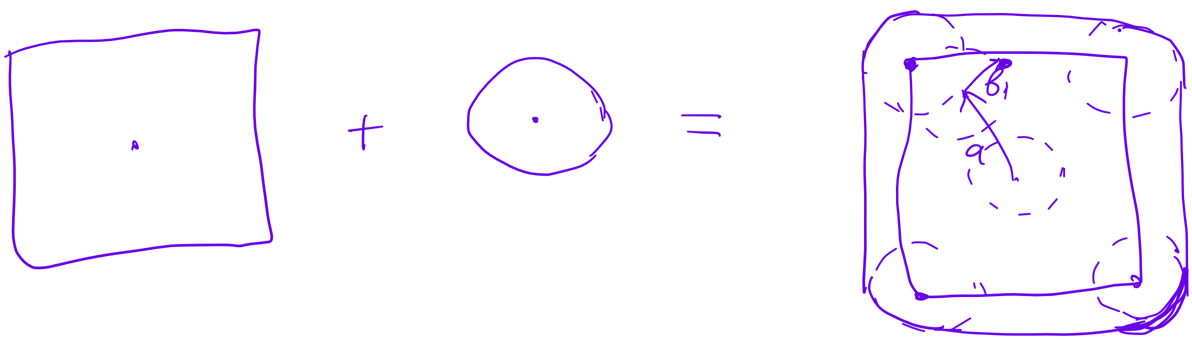
perimeter of K

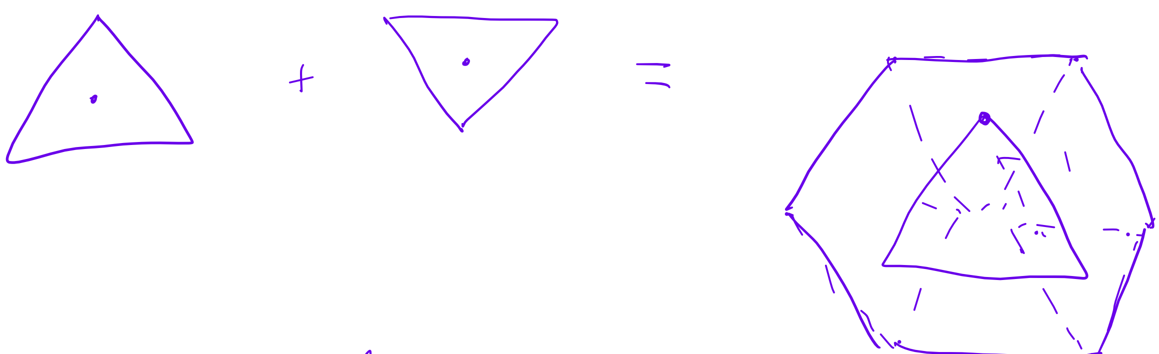
Here "$perim(K) = \lim_{\varepsilon \to 0} \frac{Vol(\varepsilon\text{-annulus of } K)}{\varepsilon}$"

<u>Def</u> (Minkowski addition of sets)

$$\begin{cases} A, B \subset \mathbb{R}^n \\ A+B = \{a+b : a \in A,\ b \in B\} \end{cases}$$

<u>Examples</u>





<u>Def</u> (formal definition of <u>perimeter</u> or surface area)

$K \subset \mathbb{R}^n$ measurable set

$$\liminf_{\varepsilon \to 0} \frac{\mathrm{Vol}_n\left((K + \varepsilon \cdot B_2^n) \setminus K \right)}{\varepsilon} =: \mathrm{perim}(K)$$

Isoperimetric inequality: $\mathrm{Vol}_n(K) = \mathrm{Vol}_n(B_2^n)$

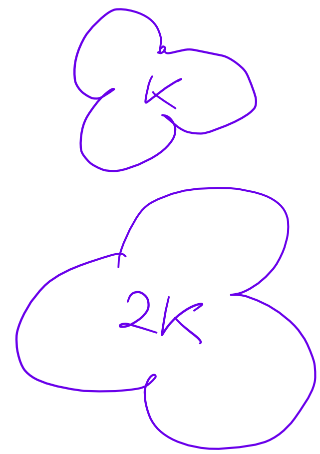
$$\Downarrow$$

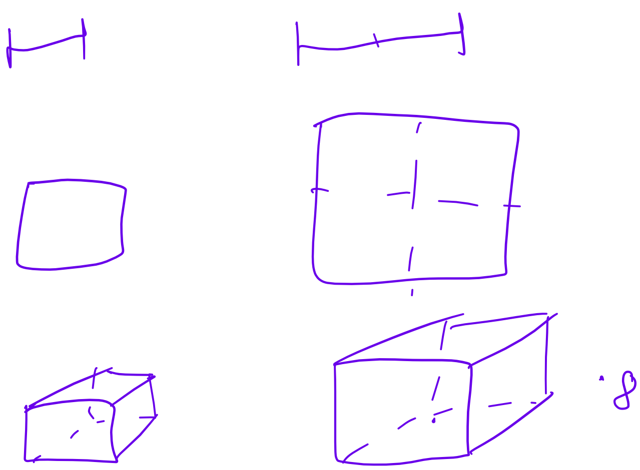
$$\mathrm{perim}(K) \geq \mathrm{perim}(B_2^n)$$

Recall an important fact

- $t > 0$ $\quad tK = \{tx : x \in K\}$

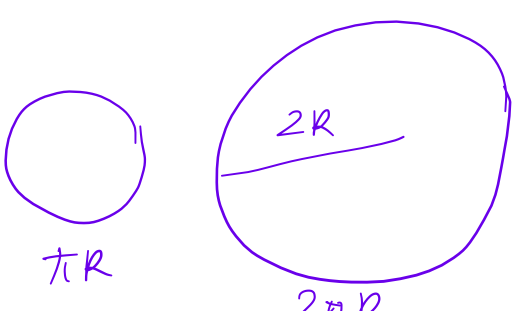
$\mathrm{Vol}_n(tK) = t^n \cdot \mathrm{Vol}_n(K)$





n-homogeneous

- $\mathrm{perim}(tK) = t^{n-1} \cdot \mathrm{perim}(K)$



Reformulation of the isoperimetric inequality

(HW)
$$\frac{\operatorname{perim}(K)}{\operatorname{Vol}_n(K)^{\frac{n-1}{n}}} \geq \frac{\operatorname{perim}(B_2)}{\operatorname{Vol}_n(B_2^n)^{\frac{n-1}{n}}}$$

for any set $K \subset \mathbb{R}^n$.

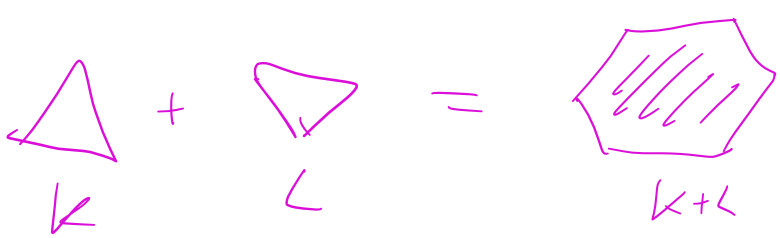
II The Brunn-Minkowski inequality

Thm $K, L \subset \mathbb{R}^n$ Borel-measurable $\Rightarrow$

$$\operatorname{Vol}_n(K+L)^{\frac{1}{n}} \geq \operatorname{Vol}_n(K)^{\frac{1}{n}} + \operatorname{Vol}_n(L)^{1/n}$$

Notation: Volume of K: $\operatorname{Vol}_n(K) = |K|$!

$$|K+L|^{1/n} \geq |K|^{1/n} + |L|^{1/n}$$



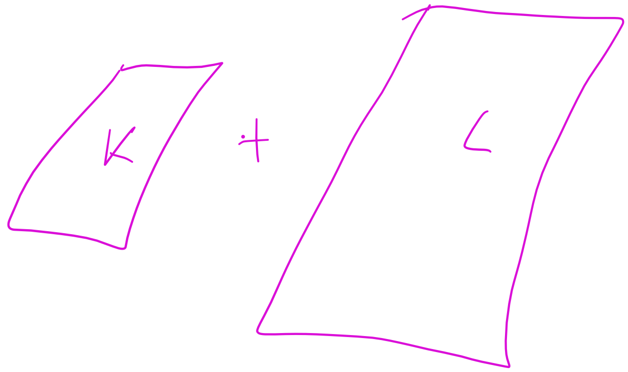
$|K+L|^{1/2} \geq |K|^{1/2} + |L|^{1/2}$

$⬡^{1/2} \geq 2\,\triangle^{1/2}$

$⬡ \geq 4 \cdot \triangle$

! Equality in the BM inequality $\Longleftrightarrow$ $K = a \cdot L$ for some $a > 0$

(HW)



Note that if $K = aL$

$$K + aL = (a+1)K$$

$$\text{LHS} = |K + aL|^{1/n} = |(a+1)K|^{1/n} = \left((a+1)^n \cdot |K|\right)^{1/n} = (a+1)\cdot |K|^{1/n}$$

$$\text{RHS} = |K|^{1/n} + |L|^{1/n} = |K|^{1/n} + \left(a^n \cdot |K|\right)^{1/n} = (a+1)|K|^{1/n}$$

① Reformulation of BM inequality

$\lambda \in [0,1]$, $K, L \subset \mathbb{R}^n$

(☆) $$|\lambda K + (1-\lambda)L|^{1/n} \geq \lambda |K|^{1/n} + (1-\lambda)|L|^{1/n}$$

In other words $|K|^{1/n}$ is concave in Minkowski addition

② Reformulation

$\lambda \in (0,1]$, $K, L \subset \mathbb{R}^n$

(☆☆) $$|\lambda K + (1-\lambda)L| \geq |K|^{\lambda} \cdot |L|^{1-\lambda}$$

Recall $a, b > 0$, $p > 0$, $\lambda \in (0,1]$

$$\left(\lambda a^p + (1-\lambda) b^p\right)^{1/p} \searrow \text{ in } p$$

$p > q > 0$

$a = |K|$
$b = |L|$

$$\left(\lambda a^p + (1-\lambda) b^p\right)^{1/p} \geq \left(\lambda a^q + (1-\lambda) b^q\right)^{1/q} \geq a^\lambda b^{1-\lambda}$$

$$\text{“} \left(\lambda a^0 + (1-\lambda) b^0\right)^{\frac{1}{0}} \text{”}$$

Therefore, $|\lambda K + (1-\lambda) L| \geq \left(\lambda |K|^{1/n} + (1-\lambda) |L|^{1/n}\right)^n$

$$\geq |K|^\lambda \cdot |L|^{1-\lambda}$$

$(\star) \Rightarrow (\star\star)$

In fact, $(\star\star) \Rightarrow (\star)$ in view of the homogeneity of the volume

(HW)

III Brunn-Minkowski $\Rightarrow$ the isoperimetric inequality

Claim
$$\frac{\text{perim}(K)}{|K|^{\frac{n-1}{n}}} \geq \frac{\text{perim}(B_2^n)}{|B_2^n|^{\frac{n-1}{n}}}$$

PROOF

Recall : perim $(K) := \liminf_{\varepsilon \to 0} \frac{|(K+\varepsilon B_2^n) \setminus K|}{\varepsilon}$

$$= \liminf_{\varepsilon \to 0} \frac{|K+\varepsilon B_2^n| - |K|}{\varepsilon} \geq$$

Brunn-Minkowski : $|K+\varepsilon B_2^n| \geq \left(|K|^{1/n} + |\varepsilon B_2^n|^{1/n}\right)^n$

$$= \left(|K|^{1/n} + \varepsilon \cdot |B_2^n|^{1/n}\right)^n$$

$$\geq \liminf_{\varepsilon \to 0} \frac{\left(|K|^{1/n} + \varepsilon |B_2^n|^{1/n}\right)^n - |K|}{\varepsilon} =$$

Newton's binomial $(a+b)^n = a^n + n a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + b^n$

$$\left(|K|^{1/n} + \varepsilon |B_2^n|^{1/n}\right)^n = |K| + n|K|^{\frac{n-1}{n}} \cdot |B_2^n|^{1/n} \cdot \varepsilon + \binom{n}{2} |K|^{\frac{n-2}{n}} \cdot |B_2^n|^{2/n} \varepsilon^2 + \dots$$

$$= |K| + n|K|^{\frac{n-1}{n}} \cdot |B_2^n|^{1/n} \cdot \varepsilon + o(\varepsilon)$$

$$= \liminf_{\varepsilon \to 0} \frac{\cancel{|K|} + n|K|^{\frac{n-1}{n}} |B_2^n|^{1/n} \cdot \varepsilon + o(\varepsilon) - \cancel{|K|}}{\varepsilon}$$

$$= n|K|^{\frac{n-1}{n}} \cdot |B_2^n|^{1/n}$$

Conclude: $$\frac{\mathrm{perim}(K)}{|K|^{\frac{n-1}{n}}} \geq n \cdot |B_2^n|^{1/n}$$

Remains to note $n|B_2^n|^{1/n} = \frac{\mathrm{perim}(B_2^n)}{|B_2^n|^{\frac{n-1}{n}}}$

Indeed, if $K = B_2^n$ we have equality in the chain above.

Alternatively can be shown (hw)

$\blacksquare$

<u>Remark</u> (answering to João)

one can also define an anisotropic perimeter
$$\mathrm{perim}_L(K) = \liminf_{\varepsilon \to 0} \frac{\mathrm{Vol}_n(K+\varepsilon L \setminus K)}{\varepsilon}$$
and also get an inequality.

WILL DISCUSS with Stephanie

IV Log-concave functions and measures

<u>Def</u> (log-concave function) $f: \mathbb{R}^n \to \mathbb{R}$

$0 \leq f$ is called log-concave if

$\log f$ is a concave function, i.e.

$\forall x, y \in \mathbb{R}^n \quad \forall \lambda \in [0,1]$

$\log f(\lambda x + (1-\lambda) y) \geq \lambda \log f(x) + (1-\lambda) \log f(y)$

$\log f(\lambda x+(1-\lambda)y) \geq \lambda \log f(x) + (1-\lambda)\log f(y)$

OR

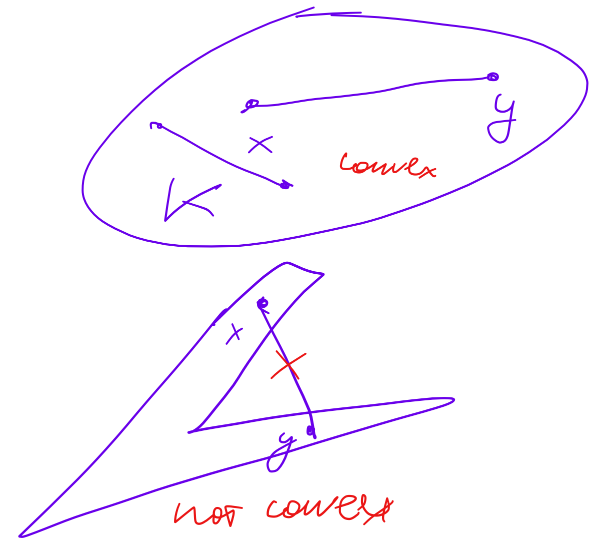
$$\boxed{f(\lambda x+(1-\lambda)y) \geq f(x)^{\lambda} f(y)^{1-\lambda}}$$

Examples

- K - convex set $\left(\forall x, y \in K \ \forall \lambda \in [0,1] \ \ \lambda x+(1-\lambda)y \in K\right)$

$f(x) = \mathbb{1}_K(x) \ominus$ is log-concave

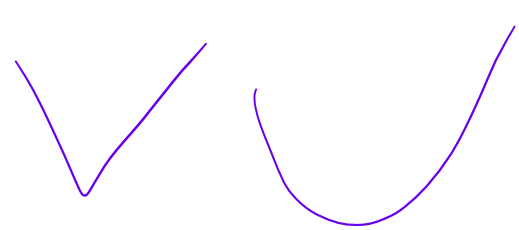
$\ominus \begin{cases} 1, & x \in K \\ 0, & x \notin K \end{cases}$



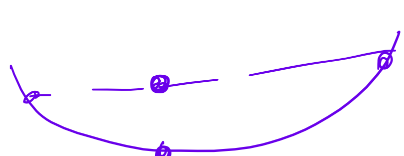
- $f(x) = e^{-|x|^2/2}$

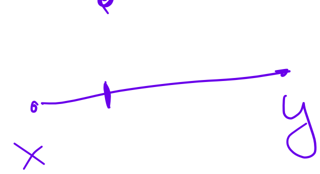
- $f(x) = e^{-V(x)}$

V - convex function



$$V(\lambda x+(1-\lambda)y) \leq \lambda V(x) + (1-\lambda)V(y)$$





Def (log-concave measure)

μ on $\mathbb{R}^n$ (measure) is called *log-concave* if $\forall K, L$ - Borel measurable sets in $\mathbb{R}^n$

$\forall \lambda \in (0,1]$

$$\mu(\lambda K + (1-\lambda)L) \geq \mu(K)^{\lambda} \cdot \mu(L)^{1-\lambda}$$

Theorem (Borel)

A measure μ on $\mathbb{R}^n$ is log-concave iff it has a density (either on $\mathbb{R}^n$ or on an affine subspace) f, and f is a log-concave function.

Remark Borell thm $\Rightarrow$ Brunn-Minkowski